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Olami-Feder-Christensen model on different networks

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Abstract. We investigate numerically the Self Organized Criticality (SOC) properties of the dissipative Olami-Feder-Christensen model on small-world and scale-free networks. We find that the small-world OFC model exhibits self-organized criticality. Indeed, in this case we observe power law behavior of earthquakes size distribution with finite size scaling for the cut-off region. In the scale-free OFC model, instead, the strength of disorder hinders synchronization and does not allow to reach a critical state.

PACS. 05.65.+b Self-organized systems – 45.70.Ht Avalanches – 89.75.Da Systems obeying scaling laws – 91.30.Bi Seismic sources (mechanisms, magnitude, moment frequency spectrum)

1 Introduction

The idea of the seismogenic crust as a self-organized complex system was introduced over the years as a possible explanation for the widespread occurrence of space-time long-range correlations in earthquakes dynamics, similar to those observed in critical phase transitions [1]. In general, the term self-organized criticality (SOC) [2] refers to the intrinsic tendency of a large class of spatially extended dynamical systems to spontaneously organize into a dynamical critical state. One signature of SOC is the presence of both a power law behavior in earthquakes size distributions and a finite size scaling for their cutoffs. Among the great number of different SOC models [3,4] developed in the last years, the OFC model [5], introduced by Olami, Feder and Christensen in 1992, has played a key role in modelling earthquakes phenomenology. However the presence of criticality in the non-conservative version of this model has been controversial since its introduction [6] and it is still debated [7,8], also in relation with the influence of topology. In literature, OFC models on different topologies have been investigated, in particular the 2D nearest neighbor lattice (NNL) model [9], annealed random neighbor (ARN) graph model and the OFC model on a quenched random (QR) graph [10]. The purpose of our work is to study the effects of small-world (SW) and scale-free (SF) topologies on the criticality of the non-conservative OFC model. These results represent a further step ahead of a project still in progress, which extends a previous paper [11]. Recent studies of earthquakes networks extracted from real data can be found in references [12–14]. In our case for the moment we do not compare with real data and we leave it for a future study.

The paper is organized in the following way. In Section 2 we review the OFC model and we point out the main reasons that have induced us to study the non-conservative OFC model on SW and SF topologies. In Section 3 we investigate the SW OFC model: in Section 3.1 we show the earthquakes size distributions for the non-conservative SW OFC model and in Section 3.2 we characterize the critical behavior of the model through the finite size scaling ansatz. Finally, in Section 4 we investigate the OFC model on a scale-free network, obtained by preferential attachment procedure [15]. Conclusions are drawn in Section 5.

2 The Olami-Feder-Christensen model

The Olami-Feder-Christensen (OFC) model [5] is defined on a discrete system of N sites (blocks or fault elements) on a square lattice, each carrying a seismogenic force (see Fig. 1). Such a force is simulated by associating to each site a real variable F_i, which initially takes a random value in the interval $(0, F_{th})$. All the forces are increased simultaneously and uniformly (mimicking a uniform tectonic loading), until one of them reaches the threshold value F_{th} and becomes unstable ($F_i \geq F_{th}$). The uniform driving is then stopped and an "earthquake" (or avalanche) starts:

$$F_i \geq F_{th} \Rightarrow \begin{cases} F_i \to 0 \\ F_{nn} \to F_{nn} + \alpha F_i \end{cases} \quad (1)$$

where "nn" denotes the set of nearest-neighbor sites of i. The parameter α controls the level of conservation of the

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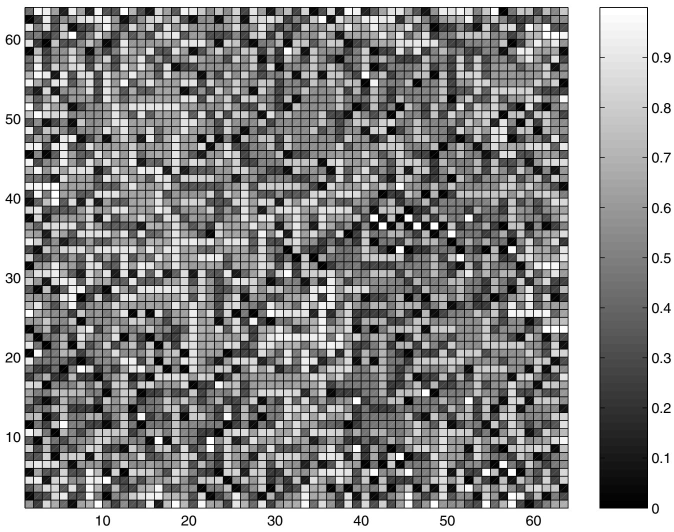

Fig. 1. Critical stress field of a 64 × 64 lattice (NNL OFC model) in the critical state.

Table 1. In this table we list the SOC properties for OFC models on different topologies: a two dimensional nearest neighbor lattice (2D NN lattice), an annealed random neighbor (ARN) graph, a quenched random (QR) graph, a quenched random (QR+2) graph with two sites with coordination 3, a small-world (SW) network and a scale-free (SF) network. We always consider these models in the case $\alpha = 0.21$ (dissipative regime) and with open boundary conditions. In particular, we report when there is power law and finite size scaling in earthquakes size distribution, according to each kind of topology.

Topology	Power law	Finite size scaling
2D NN lattice	Yes	No
ARN graph	No	No
QR graph	No	No
QR graph+2	Yes	Yes
SW network	Yes	Yes
SF network	No	No

dynamics and, in the case of a graph with fixed connectivity q, it takes values between 0 and $1/q$ ($\alpha = 1/q$ corresponding to the conservative case). The toppling rule (1) can possibly create new unstable sites, producing a chain reaction. All sites that are above threshold at a given time step in the avalanche relax simultaneously according to (1) and the earthquake is over when there are no more unstable sites in the system ($F_i < F_{th}$, $\forall i$). The uniform growth then starts again. The number of topplings during an earthquake defines its size, s, and we will be interested in the probability distribution $P_N(s)$. In the following the boundary conditions of the model will be "open", implying that $F = 0$ on the boundary sites.

At this point it is important to emphasize that the OFC model behavior strongly depends on the chosen topology. For instance, in the dissipative NNL OFC model with open boundary conditions the earthquakes size distribution is described by a power law [9,16], characterized by a universal exponent $\tau \simeq 1.8$ independent of the dissipation parameter. However, at variance with the conservative case where a full SOC behavior is observed, finite size scaling appears to be violated in the pdf cutoff-region (see Tab. 1).

In ARN OFC models [17–20], where each site interacts with randomly chosen sites instead of its nearest neighbors on the lattice, there is criticality only in the conservative case, where it becomes equivalent to a critical branching process. As soon as some dissipation is introduced, the earthquakes become localized although the mean earthquakes size diverges exponentially as dissipation tends to zero and there is no power law distribution (see Tab. 1). Actually it is interesting to point out that criticality in the OFC model on a lattice has been ascribed to a mechanism of partial synchronization [21]. In general the system shows a tendency to self-organize into a periodic state [21–23] which is frustrated by the presence of inhomogeneities such as the boundaries. In addition, inhomogeneities induce partial synchronization of the elements of the system building up long range spatial correlations and a critical state is obtained. The mechanism of synchronization requires an underlying spatial structure and therefore cannot operate in an ARN model, where each site is assigned new random neighbors at each update.

In the OFC model on a QR graph, where the choice of neighbors is not annealed but quenched and all the sites have exactly the same number of nearest neighbors q (both for $q = 4$ and $q = 6$), the dynamics organizes into a subcritical state. This is analogous to what happens in the OFC model on a NN lattice with periodic boundary conditions, where no critical behavior is observed at all [21–23]. In the QR case, in order to observe scaling in the earthquakes distribution, one has to introduce some inhomogeneities. In particular, it has been found that it is enough to consider just two sites in the system with coordination $q - 1$ [10]. When either of these sites topple according to rule (1), an extra amount αF_i is simply lost by the system. In such a way spatial correlations can develop, because the topology is quenched, there is power law in earthquakes size distribution and also finite size scaling is observed (see Tab. 1).

In this work we study the non-conservative OFC model on both a small-world and a scale-free topology.

First of all we expect that the inclusion of some inhomogeneities in the sites degree is not the unique way to obtain SOC. Indeed, as we are going to show, an alternative way is to keep fixed the sites degree and to change the topology of the underlying network, for instance by considering a small-world graph obtained by randomizing a fraction p of the links of the regular NN lattice. Here we will use the term "small-world" to refer to a rewired lattice (with fixed connectivity) with the minimum number of rewired links such that the characteristic path length L is almost as small as that one for the corresponding random graph [24,25]. As shown in the right panel of Figure 2, this is obtained already for very small values of p ($p \simeq 0.01$), much before the random graph limit ($p = 1$).

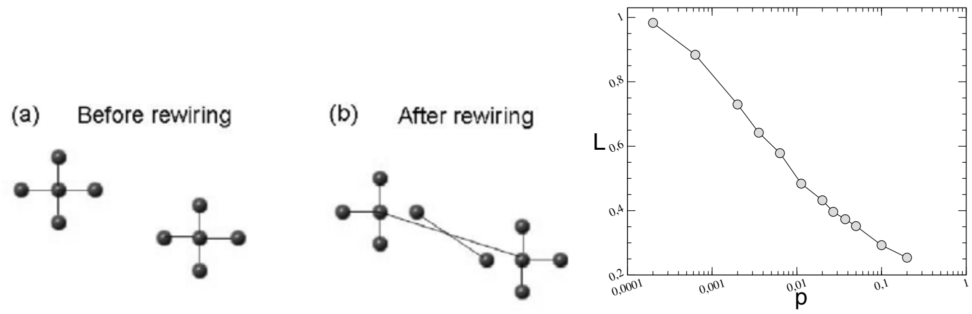

Fig. 2. On the left, a schematic picture of the rewiring procedure to interpolate between a regular and a random topology by keeping fixed and equal to 4 the degree of each site. On the right, we report the characteristic path length L vs. the rewiring probability p.

A small-world topology is expected to be a more accurate description of a real system according to the most recent geophysical observations that indicate that earthquakes correlation might extend to the long range in both time and space [26,27]. In fact, if a main fracture episode occurs, it may induce slow strain redistribution through the earth crust, thus triggering long-range as well as short-range seismic effects [28–32]. The presence of a certain percentage of long-range connections in the network takes into account the possibility that an earthquake can trigger other earthquakes not only locally but also at long distances.

On the other hand, one can consider a different kind of networks with a small L, the so called "scale-free" networks, which differ from the small-world graphs for having a power law distribution of the site degree. Scale-free networks are very common in nature and have also been used for SOC models (see Ref. [33] for sandpile dynamics on SF network) but they have not been investigated, as far as we know, in the context of OFC models. It is known that, when the connectivity is not fixed at all but only in average (as for a random graph in Ref. [10]), the strength of disorder is enough to destroy critical behavior. Thus we expect that for SF networks, where the connectivity has a power law distribution, synchronization will not take place and it will be not possible for the OFC model to reach a critical state. In the last part of the paper we will show that this is exactly what happens.

3 The OFC model on a small-world network

To investigate the effects of the small-world topology on the criticality of the non-conservative OFC model, we follow the method proposed by Watts and Strogatz to construct a network which interpolates between a square NN lattice and a quenched random graph [24,34]. We start with a two-dimensional NN square lattice in which each site is connected to its 4 nearest neighbors. The links of the lattice are rewired at random with a probability p as in the one-dimensional model of reference [24]. The main difference with respect to the original model is that for any value of p we want to keep fixed the connectivity of each site. For such a reason we have implemented a rewiring procedure as in Figure 2 in which the connections are rewired in couples. We choose a site i_1 and the edge $i_1 - i_2$ that connects site i_1 to its nearest neighbor i_2 in a clockwise sense. With probability p we decide whether to rewire this edge or to leave it in place. If the edge has to be rewired we (a) choose at random a second site j_1 and one of its edges, for instance the edge $j_1 - j_2$ connecting site j_1 to site j_2, and (b) we substitute the couple of edges $i_1 - i_2$ and $j_1 - j_2$ with the couple $i_1 - j_2$ and $j_1 - i_2$.

We repeat this process by moving over the entire square lattice considering each site in turn until one lap is completed. In such a way the limit case $p = 1$ is a QR graph with fixed connectivity (q) equal to 4. In the intermediate cases $0 < p < 1$ we can investigate the effects of an increasing number of long-range connections on the criticality of the model. Indeed, at a critical region of the parameter p between the regular ($p = 0$) and random ($p = 1$) networks, the topology produced by such a method exhibits a small-world behavior, characterized by the fact that the distance between any two sites on the graph is of the order of that for a random network and, at the same time, the concept of neighborhood is preserved, as for regular lattices. For this reason, we expect to obtain SOC in a small-world topology; the introduction of a few long-range edges create short-cuts that connect sites that otherwise would be much further apart.

3.1 Earthquakes size distributions

In our simulations the starting point for the construction of the SW network is a two-dimensional square lattice $L \times L$ with three different sizes: $L = 32, 64$ and 128; the corresponding number of sites is $N = L^2$. We have considered up to 10^9 earthquakes to obtain a good statistics for the earthquakes size distribution $P_N(s)$. In Figure 3 we report the power law distributions resulting for $N = 64^2$, $\alpha = 0.21$ (non-conservative OFC model) and for two rappresentative values of the rewiring probability p (actually we made the simulations also for many other values of p in the range [0,0.1]). In the same figure we report also the comparison with the earthquakes size distribution for the dissipative OFC model on a scale-free network (that will be discussed in Sect. 4).

All the curves can be fitted by a stretched-exponential function $P_N(s) = As^{-\tau}e^{-(s/\xi)^{\sigma}}$, where s is the size of earthquakes, ξ is the characteristic length and τ and σ

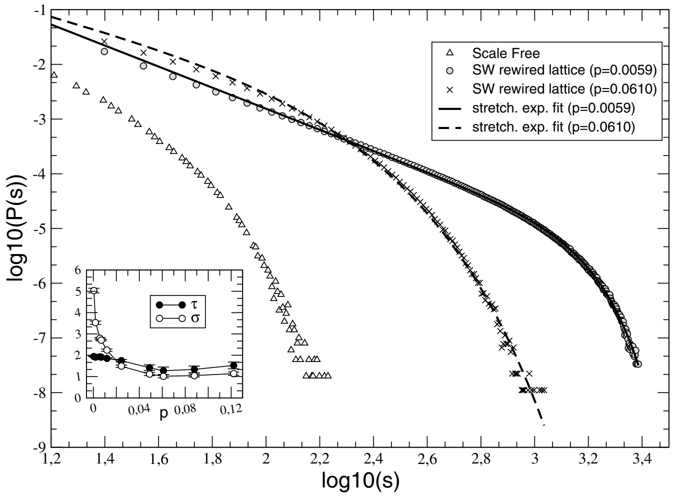

Fig. 3. Earthquakes size distributions for the non-conservative OFC model (with $\alpha = 0.21$) on rewired 64×64 lattice with two rappresentative values of rewiring probability p in the range [0,0.1]. A stretched-exponential function is used to fit the pdf's cutoffs. In the inset we plot the two exponents τ and σ as a function of the rewiring probability p. Here we report also the pdf for SF OFC model (see Sect. 4).

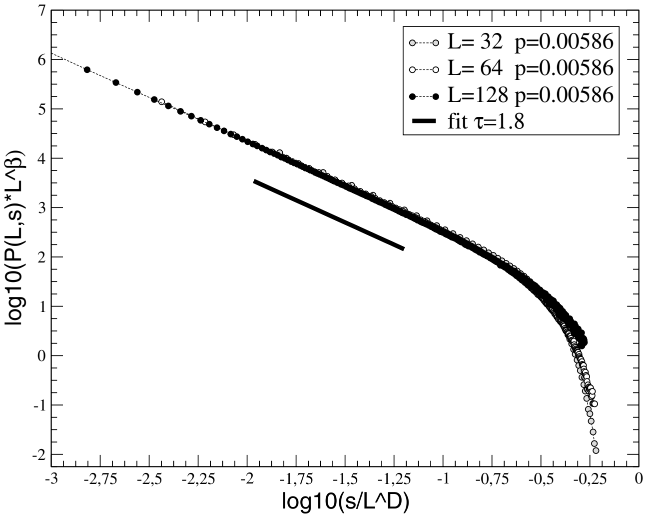

Fig. 4. Finite Size Scaling for dissipative OFC model (with $\alpha = 0.21$) on a small-world topology for three different values of N, namely $N = 32^2, 64^2, 128^2$. The critical exponent derived from the fit are $D = 2$ and $\beta \simeq 3.6$ and the rewiring probability is equal to 0.00586.

are two exponents. We notice that, increasing more and more the rewiring probability, the power law is practically lost. This can be better exploited by plotting the value of the two exponents τ and σ as a function of p in the inset in Figure 3. Indeed one can expect stretching-exponential in various cases of stochastic processes where many length scales appear. We note also that, above the value $p \simeq 0.01$ for which σ suddenly approaches 1, the power law for the pdfs progressively disappears. In the next subsection we will show that the cut-off in the earthquakes probability distribution scales with the system size (the so called finite size scaling ansatz) only around this rewiring threshold.

3.2 Finite size scaling

In order to characterize the critical behavior of the dissipative SW OFC model, a finite size scaling (FSS) ansatz is applied, i.e. $P_N(s) \simeq N^{-\beta} f(s/N^D)$ where f is a suitable scaling function and β and D are critical exponents describing the scaling of the distribution function. In Figure 4 we consider $\alpha = 0.21$ and a rewiring probability $p \simeq 0.006$. We show the collapse of $P_N(s)$ for three different values of N, namely $N = 32^2, 64^2, 128^2$. The distribution $P_N(s)$ satisfies the FSS hypothesis reasonably well with universal critical coefficients with small rewiring probability, but, increasing p, as shown in the previous subsection, there is no power law at all. The critical exponents derived from the fit of Figure 4 are $\beta \simeq 3.6$ and $D = 2$. This result is in agreement with the FSS hypothesis implying that, for asymptotically large N, $P_N(s) \sim s^{-\tau}$ with $\tau = \beta/D \simeq 1.8$.

Therefore, showing both power law behavior and FSS, the dissipative SW OFC model (in a restricted range of rewiring probability) clearly exhibits self-organized criticality (see Tab. 1). Let us point out that on the SW rewired topology the system behaves as in the compact square lattice, but the occurrence of a small amount of long-range links disseminates the earthquakes over the network and the biggest earthquake size scales with the lattice. On the other hand, if we increase the amount of long-range links above a certain threshold ($p \simeq 0.006$), the mechanism of synchronization is corrupted and the scaling behavior disappears.

4 The OFC model on a scale-free network

Finally we investigate criticality of the non-conservative OFC model on a scale-free network. It is an example of network displaying a small characteristic path length and a power-law distribution $p(k) \sim k^{-\gamma}$ in the node connectivity k (degree). By using the preferential attachment growing procedure introduced by Barabási and Albert [15], we start from $m + 1$ all to all connected nodes and at each time step we add a new node with m links. These m links point to old nodes with probability $p_i = \frac{q_i}{\sum_j q_j}$, where q_i is the degree of the node i. This procedure allows a selection of the γ exponent of the power law scaling in the degree distribution with $\gamma = 3$ in the thermodynamic limit ($N \longrightarrow \infty$). Here we consider a scale-free network with $\gamma = 3$ and $N = 1000$.

In this case, the toppling rule in equation (1) must be modified to take into account that different sites have a different coordination number q_i. Each site consequently has a different α_i, which we determined by requiring that the total fraction $\tilde{\alpha}$ of the force transferred from the unstable site to the nearest-neighbor sites is constant in the system, i.e., $\alpha_i = \tilde{\alpha}/q_i$; here we consider the case $\tilde{\alpha} = 0.84$.

We have found that there is no criticality in the system since there is no power law in the earthquakes size distribution, as shown in Figure 3. As previously observed and in agreement with previous investigations [9,35,36], this result indicates that if the disorder is too strong the critical signatures disappear and the SOC behavior is destroyed (see Tab. 1).

5 Conclusions

In conclusion, in this paper we have investigated the dissipative OFC model on small-world and scale-free networks.

We have shown that, at variance with OFC models on other topologies which are critical only in the conservative case, the dissipative small-world OFC model clearly reaches a critical state characterized by power law behavior of earthquakes size distribution with finite size scaling of cut-offs. Indeed, in a lattice with a small number of rewired links the underlying spatial structure allows partial synchronization of distant blocks of the system. We think that this process could reproduce the long-range earthquakes dynamical correlations in the earth crust, according to the most recent geophysical observations.

On the other hand, on a scale-free topology we do not observe SOC properties. We expected this behavior because the connectivity is not fixed; so the dynamics is not synchronized, the disorder is too strong and the critical state is destroyed. As future directions, it seems interesting to better investigate the influence of topology and the role of disorder on the self-organized criticality properties of the OFC models.

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