Anomalous Fluctuation Relations

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Outline

- **'Normal' fluctuation relations:**

  motivation and warm-up

- **Gaussian stochastic dynamics:**

  check transient fluctuation relations for generalized (correlated) Langevin dynamics

- **Relations to experiments:**

  glassy dynamics and cell migration

- **Other anomalous dynamics:**

  normal / anomalous fluctuation relations for Lévy flights and time-fractional kinetics (CTRW)

Motivation: Fluctuation relations

Consider a (classical) particle system evolving from some initial state into a nonequilibrium steady state.

Measure the probability distribution $\rho(\xi_t)$ of entropy production ξ_t during time t:

$$\ln \frac{\rho(\xi_t)}{\rho(-\xi_t)} = \xi_t$$

Transient Fluctuation Relation (TFR)

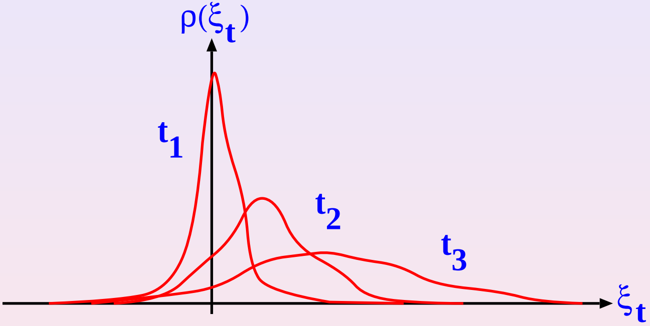
Evans, Cohen, Morriss (1993); Gallavotti, Cohen (1995)

why important? of *very general validity* and

1. generalizes the Second Law to small noneq. systems
2. connection with fluctuation dissipation relations
3. can be checked in experiments (Wang et al., 2002)

Fluctuation relation and the Second Law

meaning of TFR in terms of the Second Law:



$t_1 < t_2 < t_3$

$$\boxed{\rho(\xi_t) = \rho(-\xi_t)\exp(\xi_t)} \geq \rho(-\xi_t)\ (\xi_t \geq 0) \ \Rightarrow <\xi_t> \geq 0$$

sample specifically the tails of the pdf (large deviation result)

Fluctuation relation for Langevin dynamics

warmup: check TFR for the overdamped Langevin equation

$$\dot{x} = F + \zeta(t) \qquad \text{(set all irrelevant constants to 1)}$$

with constant field F and Gaussian white noise $\zeta(t)$.

entropy production ξ_t is equal to (mechanical) work $W_t = Fx(t)$ with $\rho(W_t) = F^{-1}\varrho(x,t)$; remains to solve corresponding Fokker-Planck equation for initial condition $x(0) = 0$:

the position pdf is Gaussian,

$$\varrho(x,t) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{(x - <x>)^2}{2\sigma_x^2}\right)$$

straightforward:

(work) TFR holds if $< x >= \sigma_x^2/2$

and $\exists$ fluctuation-dissipation relation 1 **(FDR1)** $\Rightarrow$ **TFR**

see, e.g., van Zon, Cohen, PRE (2003)

Gaussian stochastic dynamics

goal: check TFR for Gaussian stochastic processes defined by the (overdamped) **generalized Langevin equation**

$$\int_0^t dt' \dot{x}(t') K(t-t') = F + \zeta(t)$$

e.g., Kubo (1965)

with Gaussian noise $\zeta(t)$ and memory kernel $K(t)$

such dynamics can generate anomalous diffusion:

$$\sigma_x^2 \sim t^\alpha \text{ with } \alpha \neq 1 \ (t \to \infty)$$

examples of applications: polymer dynamics (Panja, 2010); biological cell migration (Dieterich et al., 2008)

TFR for correlated internal Gaussian noise

consider two generic cases:

1. **internal Gaussian noise** defined by the **FDR2**,

$$<\zeta(t)\zeta(t')>\sim K(t-t')\ ,$$

with non-Markovian (correlated) noise; e.g., $K(t)\sim t^{-\beta}$

solving the corresponding generalized Langevin equation in Laplace space yields

FDR2 $\Rightarrow$ ‘FDR1’

and since $\rho(W_t)\sim\varrho(x,t)$ is Gaussian

‘FDR1’ $\Rightarrow$ TFR

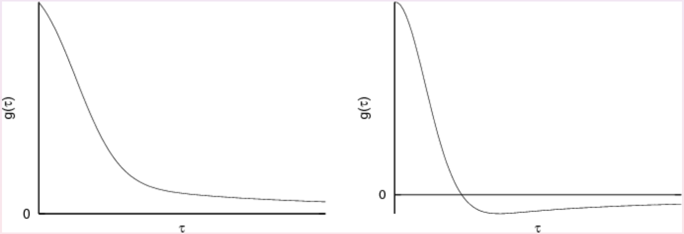
for correlated **internal Gaussian noise** $\exists$ TFR

Correlated external Gaussian noise

2. external Gaussian noise for which there is **no FDR2**, modeled by the (overdamped) generalized Langevin equation

$$\dot{x} = F + \zeta(t)$$

consider two types of Gaussian noise correlated by
$g(\tau) = <\zeta(t)\zeta(t')>_{\tau=t-t'} \sim (\Delta/\tau)^{\beta}$ for $\tau > \Delta$, $\beta > 0$:



persistent **anti-persistent**

it is $<x> = Ft$ and $\sigma_x^2 = 2\int_0^t d\tau (t-\tau) g(\tau)$

TFRs for correlated external Gaussian noise I

persistent noise:

results for σ_x^2 and the fluctuation ratio $R(W_t) = \ln \frac{\rho(W_t)}{\rho(-W_t)}$

- $0 < \beta < 1$:

superdiffusion $\sigma_x^2 \sim t^{2-\beta}$ with anomalous TFR $R \sim \frac{W_t}{t^{1-\beta}}$

- $\beta = 1$:

weak superdiffusion $\sigma_x^2 \sim t \ln\left(\frac{t}{\Delta}\right)$ with weakly anomalous TFR $R \sim W_t / \ln\left(\frac{t}{\Delta}\right)$

- $1 < \beta < \infty$:

normal diffusion $\sigma_x^2 \sim 2Dt$ with $D = \int_0^\infty d\tau g(\tau)$ and anomalous (generalized) TFR $R \sim \frac{W_t}{D}$

TFRs for correlated external Gaussian noise II

antipersistent noise:

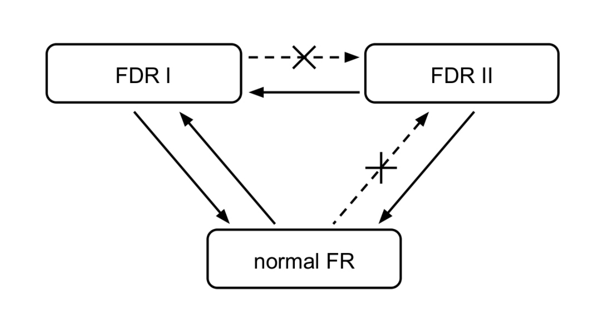
$\int_0^\infty d\tau g(\tau) > 0$ yields normal diffusion with a generalized TFR

for $t \gg \Delta$; for 'pure' antipersistent case with $\int_0^\infty d\tau g(\tau) = 0$:

- The regime $0 < \beta < 1$ does not exist (spectral density <0)
- $1 < \beta < 2$:
  subdiffusion $\sigma_x^2 \sim t^{2-\beta}$ with anomalous TFR $R \sim W_t t^{\beta-1}$
- $\beta = 2$:
  weak subdiffusion $\sigma_x^2 \sim \ln(t/\Delta)$ with anomalous TFR
  $R \sim W_t t / \ln(t/\Delta)$
- $2 < \beta < \infty$:
  localization $\sigma_x^2 = const.$ with anomalous TFR $R \sim W_t t$

FDR and TFR

relation between TFR and FDR I,II for **correlated Gaussian stochastic dynamics**: ('normal FR'= conventional TFR)



in particular:

FDR2 $\Rightarrow$ **FDR1** $\Rightarrow$ **TFR**

$\nexists$ **TFR** $\Rightarrow$ $\nexists$ **FDR2**

Relations to experiments: A glassy lattice gas

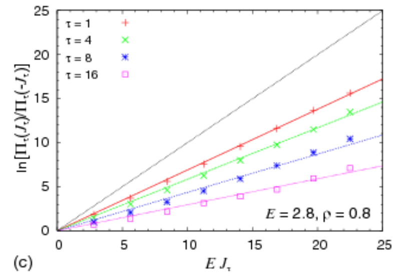
$$R(W_t) = \ln \frac{\rho(W_t)}{\rho(-W_t)} = \mathbf{f}_\beta(\mathbf{t}) W_t$$

means by plotting R for different t the slope should change.

example 1:

computer simulations for glassy lattice gas with external field E

Sellitto, PRE (2009)

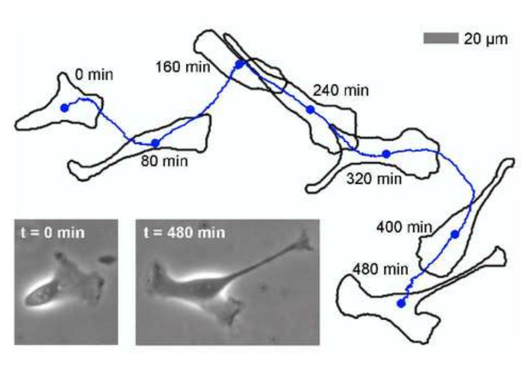


similar simulation results for three other models exhibiting glassy dynamics: Crisanti et al., PRL (2013)

Biological cell migration

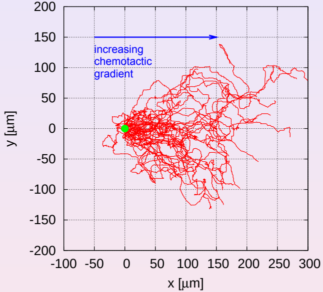
example 2:

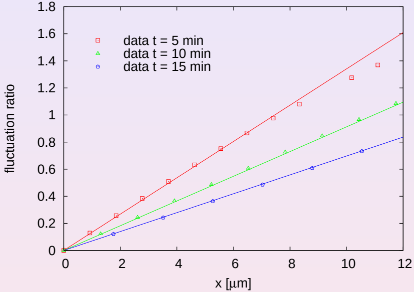
single biological cell crawling on a substrate; trajectory recorded with a video camera (Dieterich et al., 2008)



Cell migration under chemical gradients

experiments on **murine neutrophils** under **chemotaxis**:





Dieterich et al. (2013)

- linear drift in the direction of the gradient, $< x(t) >\sim t$
- $\sigma_x^2 \sim t^\beta$ with $\beta > 1$ (long t): $\nexists$ FDR1
- some relation to the generalized Langevin equation with external noise and $0 < \beta < 1$ discussed before

TFR for Lévy flights

Second type of anomalous dynamics: consider the Langevin equation

$$\dot{x} = F + \zeta(t)$$

with white Lévy noise $\rho(\zeta) \sim |\zeta|^{-1-\alpha} \; (\zeta \to \infty)$, $0 \leq \alpha < 2$

examples of applications: fluid dynamics (Solomon et al., 1993); Lévy flights for light (Barthelemy, 2008)

by solving the corresponding Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = -F \frac{\partial \rho}{\partial x} + \frac{\partial^\alpha \rho}{\partial |x|^\alpha}$$

with Riesz fractional derivative in Fourier space

$$\mathcal{F}\{\partial^\alpha \rho / \partial |x|^\alpha\} = -|k|^\alpha \mathcal{F}\{\rho\}$$

and using the scaled variable $w_t = W_t/(F^2 t)$ we recover

$$\lim_{w_t \to \pm\infty} \frac{\rho(w_t)}{\rho(-w_t)} = 1$$ Touchette, Cohen, PRE (2007)

i.e., large fluctuations are *equally possible*

TFR for time-fractional kinetics

Third type of anomalous dynamics: via subordinated Langevin equation $\frac{dx(u)}{du} = F + \zeta(u) \quad , \quad \frac{dt(u)}{du} = \tau(u)$
with Gaussian white noise $\zeta(u)$ and white Lévy stable noise $\tau(u) > 0$; leads to the time-fractional Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} \left[-\frac{\partial F}{\partial x} + \frac{\partial^2}{\partial x^2} \right] \rho$$

with Riemann-Liouville fractional derivative
$\frac{\partial^\gamma \rho}{\partial t^\gamma} = \frac{\partial^m}{\partial t^m} \left[\frac{1}{\Gamma(m-\gamma)} \int_0^t dt' \frac{\rho(t')}{(t-t')^{\gamma+1-m}} \right]$ for $m - 1 < \gamma < m \, , \, m \in \mathbb{N}$
and $\frac{\partial^\gamma \rho}{\partial t^\gamma} = \frac{\partial^m \rho}{\partial t^m}$ for $\gamma = m$, which **preserves** a generalized **FDR1**

examples of applications: photo current in copy machines (Scher et al., 1975) and related systems modeled by *Continuous Time Random Walk theory* (Metzler, Klafter, 2004)

for this dynamics we recover the conventional TFR

Summary

- TFR tested for three fundamental types of **anomalous stochastic dynamics**:
  1. correlated Gaussian stochastic dynamics:

     $$\boxed{\textbf{FDR2} \Rightarrow \textbf{FDR1} \Rightarrow \textbf{TFR}}$$

     TFR holds for *internal* noise, violations for *external persistent / anti-persistent* noise
  2. strong violation of TFR for space-fractional (Lévy) dynamics
  3. TFR holds for time-fractional (CTRW) dynamics
- anomalous TFRs appear to be important for **glassy dynamics**: cf. computer simulations on various glassy models and experiments on (‘gel-like’) cell migration

Open questions

- derive anomalous Jarzynski, Crooks, Seifert relations
- derive nonlinear response relations for anomalous dynamics
- compare anomalous fluctuation relations to experiments

References

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- A.V. Chechkin, RK, J. Stat. Mech. L03002 (2009)
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