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SCHOOL OF MATHEMATICAL SCIENCE

MSc Project

DETERMINISTIC CHAOS

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Abstract

In this project I examine the existing definitions of deterministic chaos, which in particular are due to work by Devaney,Wiggins, Lyapunov and Li and Yorke. The project starts with some basic introductory definitions before I explain the main ingredients of chaos: transitivity, denseness of periodic points, sensitive dependence on initial conditions and Lyapunov exponent. Then I illustrate by examples the chaotic behaviour by some simple but interesting maps. On this basis, crosslinks between the different chaotic ingredients are explained in a precise manner. After that I define topological mixing, topological blending, topological conjugacy and topological and metric entropy. The project finishes with a conclusion of the topics described above, and my own ideas and thoughts will be summarized.

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DETERMINISTIC CHAOS

1. Introduction

The use of the word chaos in Dynamical systems was introduced by Li and Yorke in [7]. In this paper Li and Yorke proved that for a map on the real line which has a point with period three there exists an uncountable scrambled set. The existence of this uncountable scrambled set is taken as a definition of chaos. A well-known definition of chaos is given by Devaney in [3], whose main ingredients are topological transitivity, denseness of periodic points and sensitive dependence on initial conditions .It was widely understood that sensitive dependence on initial conditions was the main ingredient in Devaney's chaos but in [1] it was proved that it was a redundant hypothesis since it was implied from the other two conditions. Other famous definitions of chaos are given by Wiggins and Lyapunov. Also chaos can be characterized in terms of the metric and topological entropy. Topological entropy is related to Li-Yorke's chaos and metric entropy to Lyapunov's chaos.

The project starts with some fundamental elementary definitions basically from point-set topology and from dynamical systems. Section 2.2 gives the main ingredients of chaos, namely topological transitivity, dense set of periodic points and sensitive dependence on initial conditions. Section 2.3 provides three basic definitions of chaos -Devaney's chaos, Wiggins' chaos and Lyapunov's chaos. Also in the same section some theorems and propositions are given relating these three definitions of chaos and their ingredients. The next section examines the chaotic behaviour- with respect to these three chaotic definitions - of some well-known maps such as the Bernoulli shift and the Tent map. Section 3.1 introduces the topological mixing and topological blending (strong and weak) as an alternative of topological transitivity in Devaney's chaos. As it will be proved both these two conditions are not equivalent to transitivity. In the same manner section 3.2 introduces expansivity as an alternative of sensitivity. In the next section some more crosslinks between chaotic ingredients are given. Section 4.1 examines chaos with respect to discontinuity. As I will prove a transitive map with a point of discontinuity implies denseness of periodic points. In

section 4.2 I introduce the scrambled sets and I define chaos in the sense of Li and Yorke.
Section 4.3 is devoted to the important concept of topological and metric entropy. As I mentioned in this section positivity of these two entropies implies chaos. In section 4.4 I study the behaviour of some chaotic ingredients with respect to topological conjugation and finally in section 4.5 I give an overview of what I have already talked about and I give some open questions.
The standpoint of this project is mostly topologic and I don't discuss any topic in terms of ergodicity or measure theory. This does not imply that such matters are without interest merely they are outside of my scope.

2. Three basic definitions of chaos

2.1 Preliminaries

In this part of my project I will give some introductory definitions that will be used several times later. These definitions can be found in every book of Dynamical Systems.

Definition 2.1.1 Consider the continuous function $f: X \to X$. The Dynamical System defined by f takes the form $x_{n+1} = f(x_n)$ and is written as (X, f).Such functions that describe Dynamical Systems are called maps.

Definition 2.1.2 Let X, Y be subsets of a metric space Z such that $X \subset Y$. We say that X is dense in Y if $\overline{X} = Y$, i.e $\forall\ x \in Y$, $\forall \varepsilon > 0$ $N_\varepsilon(x)$ contains a point in X($\overline{X}$ is defined to be the closure of X)

Definition 2.1.3 Consider the continuous map f: $X \to X$. A point $x \in X$ is said to be a fixed point for f if $f(x)=x$. The set of fixed points of f is denoted by $Fix(f) = \{x \in X : f(x) = x\}$.
If $f^n(x)=x$ for some $n \in \mathbb{N}$ then x is a periodic point of f with period n. The set of the periodic points of f with period n is denoted by $Per_n(f) = \{x \in X : f^n(x) = x\}$.

The orbit of a point x is the set of the points x, $f(x)$, $f^2(x)$,..... and is denoted by $O(x) = \{x, f(x), f^2(x),..\}$ (in the case where f is not invertible).

If f is invertible then the orbit of x is $O(x) = \{f^{-1}(x), x, f(x),...\}$. The set of all these iterations of a periodic point form a periodic orbit.

Definition 2.1.4 Let X and Y be metric spaces. We say that the metric space X is compact if every open cover of X has a finite subcover, i.e. if $\{I_i\}_{i\in I}$ is a collection of open sets of X such that $X \subset \bigcup_{i\in I} Ii$ then we have that $X \subset \bigcup_{i=1}^{n} Ii$.

Also compact spaces on the real line can be thought as closed and bounded intervals.

Definition 2.1.5 A sequence $\{x_n\}_{n\in\mathbb{N}} \in X$,where X is a metric space equipped with a metric d, is Cauchy if $\forall \varepsilon > 0 \quad \exists \; n = n(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon \quad \forall m > n$. Then X is said to be complete if every Cauchy sequence on X converges(that is ,has a limit which is an element of X). Also X is said to be separable if it has a countable subset which is dense in X.

Now if Y with is another metric space with metric γ and f: $X \to Y$ map then f is said to be an isometry if f preserves distances, i.e. $\forall x, y \in X \quad \gamma(f(x), f(y) = d(x, y)$

Definition 2.1.6 A homeomorphism $h : X \to Y$ is a continuous and bijective map with a continuous inverse (continuity of the inverse is automatically satisfied if X is compact space).

Definition 2.1.7 Consider the metric spaces X and Y and let $f : X \to X$ and g: $Y \to Y$ be continuous maps. The maps f and g are said to be topologically conjugate if there exists a homeomorphism $h : X \to Y$ such that $h \circ f(x) = g \circ h(x) \quad \forall \; x \in X$

i.e the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ h\downarrow & & \downarrow h \\ Y & \xrightarrow{g} & Y \end{array}$$

commutes. A homeomorphism satisfying this condition is called a topological conjugacy.

Definition 2.1.8 Consider the continuous and differentiable map $f: X \to X$ on a metric space *X*. Then the map *f* is said to be expanding if $|f'(x)| > 1 \quad \forall \ x \in X$.

Note: Sometimes we relax the assumptions of this definition and we only need the map to be continuous and differentiable except for a finite number of points.

2.2 Chaos ingredients

In this section I will define and explain the main ingredients of chaos: topological transitivity, denseness of periodic points and sensitive dependence on initial conditions.

Definition 2.2.1 Consider the metric space *X* and the continuous map $f: X \to X$. We say that *f* is topological transitive if for every pair of non-empty open sets *U* and *V* in *X* there exists a positive integer *k* such that $f^k(U) \cap V \neq \varnothing$.

Another famous definition of transitivity is the following:

Definition 2.2.2 We say that the map *f* is topologically transitive if $\exists x \in X$ such that its orbit $\{f^n(x) \mid n \geq 0\}$ is dense in *X*, that is, $\overline{\{f^n(x) \mid n \geq 0\}} = X$.

These two definitions of topological transitivity are clearly not equivalent:

Example 2.2.3 We consider the continuous map $f: X \to X$ where $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ equipped with the metric $d = |x - y| \quad \forall x, y \in X$. The map *f* is defined by $f(0)=0$ and $f(1/n) = 1/(n+1)$ for $n = 1,2,3,\ldots$. Then if we choose $U = \{1/2\}$ and $V = \{1\}$ then *f* does not satisfy the definition 2.2.1.

Now we can observe that the point $x = 1$ has a dense orbit in X so the definition 2.2.2 is satisfied and so it's not equivalent with 2.2.1.

According to [24] these two definitions are equivalent when X is a compact metric space. Also in [22] we can find the following proposition relating these two definitions .The proof is omitted and can be found also in [22].

Proposition 2.2.4 Consider the continuous map $f: X \to X$ where X is a complete separable space with no isolated points. Then the following are equivalent:

1) f is a topologically transitive map, that is,it has a dense orbit.
2) f has a dense positive semiorbit.
3) $\forall$ U, V non empty subsets of X there exists an integer n such that $f^n(U) \cap V \neq \varnothing$
4) $\forall U, V$ non empty subsets of X there exists a natural number n such that $f^n(U) \cap V \neq \varnothing$.

Comments:

1) A topologically transitive map has points which eventually move under iteration from one arbitrary neighbourhood to any other. Consequently the Dynamical System cannot be broken down or decomposed into two disjoint open sets which do not interact under f, i.e they are invariant under the map f (a set $A \subset X$ is invariant under f if $f(A) \subset A$).
2) Topological transitivity is a purely topologic condition.
3) If the orbit of every point $x \in X$ is dense in X then the map f is said to be minimal.

Definition 2.2.5 Consider the metric space X equipped with the metric d and the continuous map $f: X \to X$. We say that the map f exhibits sensitive dependence on initial conditions if $\exists$ $\delta > 0$- called the sensitivity constant of f –such that for any $x \in X$ and any open neighbourhood $N_\varepsilon(x)$ of x for some $\varepsilon > 0$ there exists a point $y \in N_\varepsilon(x)$ and $n \geq 0$ such that $d(f^n(x), f^n(y)) \geq \delta$.

Comments

1) A map satisfying the property of sensitivite dependence on initial conditions has points in $N_\varepsilon(x)$ which eventually separate from x by at least a distance δ under iteration of f.
2) From definition 2.2.5 we can see that not all points in the open neighbourhood $N_\varepsilon(x)$ of x eventually separate from x under iteration, but there is at least one such point in every open neighbourhood.
3) Sensitivity is a metric property since it depends on the metric of the space.
4) The sensitivity constant δ does not depend on x,nor on ε but only on the Dynamical System (X,f).

Convention: Throughout this paper "transitivity" will always mean "topological transitivity" and "sensitivity" will always mean "sensitive dependence on initial conditions".

2.3 **Defining chaos**

In this section of my project I am going to give some basic definitions of chaos such as Devaney's, Wiggins' and Lyapunov's chaos. All the background of these definitions and theorems mentioned here will be covered and explained precisely.

2.3.1 Devaney' s definition of chaos: Let $f:X\to X$ be a continuous map and X be a metric space. Then f is said to be chaotic according to Devaney or D-chaotic if:

1) f is topologically transitive.
2) The periodic points of f are dense in X.
3) f exhibits sensitive dependence on initial conditions.

In [1] Banks et. al proved that sensitivity is a redundant hypothesis in Devaney's chaos because it is implied by transitivity and density of periodic points. So on the following theorem I am giving this result together with its proof:

Theorem 2.3.2 Let $f:X \to X$ be a continuous map where X is a metric space. Then if f is topologically transitive and has dense periodic points then f exhibits sensitive dependence on initial conditions.

Proof: We can find a positive number δ_0 such that there exists a periodic point $q \in X$ the orbit of which is at a distance of at least $\delta_0/2$ from every $x \in X$. Now we obtain two distinct periodic orbits of two periodic points with no common points, say p_1 and p_2 and we let the distance between these orbits be δ_0. I will prove that f exhibits sensitive dependence on initial conditions and its sensitivity constant is $\delta = \delta_0/8$. Next we take an arbitrary point $x \in X$ and choose an open neighbourhood of x, say $N_\varepsilon(x)$. We consider now the ball $B_\delta(x)$ with radius δ and centre x. Because f has dense periodic points we can find such a periodic point p that lies in the set $N_\varepsilon(x) \cap B_\delta(x)$ with period we call n. Now it is clear that exists a periodic point q such that its orbit is at least at a distance of 4δ from the point x. Then the set $V := \bigcap_{i=0}^{n} f^{-i}(B_\delta(f^i(q))) \neq \varnothing$ because $q \in V$ and it is open.

By the transitivity of f $\exists$ $k \in \mathbb{N}$ and $y \in U$ such that $f^k(y) \in V$.

Now let $j := \left\lfloor \frac{k}{n} \right\rfloor + 1$ where $\left\lfloor \frac{k}{n} \right\rfloor$ denotes the integer part of $\frac{k}{n}$. Then we have that $1 \leq nj - k \leq n$.

So we can write $f^{nj}(y) = f^{nj-k}(f^k(y) \in f^{nj-k}(V) \subseteq B_\delta(f^{nj-k}(q))$. Using the triangle inequality and the fact that $f^{nj}(p) = p$ we get

$d(f^{nj}(p), f^{nj}(y)) = d(p, f^{nj}(y) \geq d(x, f^{nj-k}(q)) - d(f^{nj-k}(q), f^{nj}(y)) - d(p,x)$
$\geq 4\delta - \delta - \delta = 2\delta$ because $p \in B_\delta(x)$ and

$f^{nj}(y) = f^{nj}(y) = f^{nj-k}(f^k(y) \in f^{nj-k}(V) \subseteq B_\delta(f^{nj-k}(q))$. Finally using the triangle inequality again either $d(f^{nj}(x), f^{nj}(y)) > \delta$ or $d(f^{nj}(x), f^{nj}(p)) > \delta$ and the proof is completed.

Remark: In [16] it is proved that a continuous map with dense periodic points and sensitive dependence on initial conditions doesn't need to be transitive. This is proved by a counter example (see example 2.4.3).
Also in the same paper it is proved that a continuous and transitive map with sensitive dependence on initial conditions does not need to have dense periodic points (see example 2.4.2).
P. Touchey in [15] proved that a Dynamical System defined on a metric space X is D-chaotic if and only if for every pair of open sets in X there exists a periodic orbit which visits both sets.

Now it is interesting to see what is happening in the case where the metric space X which the map f is defined is a finite or infinite interval on the real line. In [8] it is proved that in this case transitivity implies Devaney's chaos. First of all I will give a lemma that it is used to prove this result but I will not give the proof. The proof can be found also in [8]. The proposition clearly holds when I is a compact interval. In [4] it is proved the same result as in proposition 2.3.4 for compact spaces but its proof contains non trivial results.

Lemma 2.3.3 Consider the interval I (finite or infinite) and let $f:I\to I$ be a continuous map. Also let $J\subset I$ be an interval with no periodic points of f and z, $f^n(z)$ and $f^m(z)\in J$ with $0<m<n$. Then either $z<f^m(z)<f^n(z)$ or $z>f^m(z)>f^n(z)$.

Proposition 2.3.4 Let I be an interval and $f:I\to I$ be a continuous map. If f is topologically transitive then f exhibits sensitive dependence on initial conditions and has a dense set of periodic points.

Proof: From theorem 2.3.2 we only need to prove that f has dense periodic points. If f has no dense periodic points it is clear that there exists an interval with no periodic points, say $J\subset I$. Now we choose a point x in J, an open neighbourhood of x say $N_\varepsilon(x)$ for some $\varepsilon>0$ and an open interval $E\subset J\setminus N_\varepsilon(x)$. Since f is transitive in I there exists a positive integer m such that $f^m(N_\varepsilon(x))\cap E\neq\varnothing$ and $f^m(y)\in E\subset J$ for some y in J. Now because f is continuous we can find an open neighbourhood U

of y such that $f^m(U) \cap U = \varnothing$. Also we can find a number $m<n$ and a point z in U with $f^n(z) \in U$. But then we have a contradiction from lemma 2.3.3 because then $0<m<n$ and $z, f^n(z) \in U$ since $f^m(z) \notin U$. So the periodic points of f are dense in I.

Remark:

1) For a map on an interval the only condition that should be checked is to be D-chaotic is transitivity.
2) The proposition does not hold for dimension higher than one or on the unit circle since the ordering on $\mathbb{R}$ is used in an essential way.

It is interesting to check if proposition 2.3.4 is valid for continuous maps on the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. According to [24] we have the following theorem:

Theorem 2.3.5 Consider the continuous map f: $S^1 \to S^1$. Then if f is transitive and f has at least one periodic point, then the periodic points of f are dense in S^1 and so f is D-chaotic.

2.3.6 Wiggins' definition of chaos [28]: Let $f: X \to X$ be a continuous map and X be a metric space. Then the map f is said to be chaotic according to Wiggins or W-chaotic if:

1) f is topologically transitive.
2) f exhibits sensitive dependence on initial conditions .

Before giving the third definition of chaos first I will give the definition of the Lyapunov exponent which is a number λ measuring the exponential rate at which nearby orbits are moving apart.

Definition 2.3.7 Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous and differentiable map. Then $\forall x \in \mathbb{R}$ we define the (local) Lyapunov exponent of x say $\lambda(x)$ as

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log|f'(x_i)| \quad \forall x_i \in \mathbb{R}$$

Motivation : Consider the map $x_{n+1} = f(x_n)$ and let the points x_0 and x_0' be originally displaced by $\delta = |x_0' - x_0|$. Then after n iterations of the map we get

$$\delta x_n = |x_n' - x_n| = |f^n(x_0 + \delta) - f^n(x_0)| =: \delta e^{n\lambda(x_0)} \qquad (1)$$

in the limits $\delta \to 0$ and $n \to \infty$.If we solve the last relation with respect to $\lambda(x_0)$ we get

$$\lambda(x_0) = \lim_{n\to\infty}\lim_{\delta\to 0}\frac{1}{n}\log\left|\frac{f^n(x_0+\delta) - f^n(x_0)}{\delta}\right| = \lim_{n\to\infty}\frac{1}{n}\log\left|\frac{df^n(x_0)}{dx}\right| = \lim_{n\to\infty}\frac{1}{n}\log\left|\prod_{i=0}^{n-1} f'(x_i)\right| =$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log|f'(x_i)|.$$

2.3.8 Lyapunov's definition of chaos [21]: Consider the continuous and differentiable map $f: \mathbb{R} \to \mathbb{R}$ .Then f is said to be chaotic according to Lyapunov or L-chaotic if:

1) f is topologically transitive.
2) f has a positive Lyapunov exponent.

Remark:

In a set of positive measure the Lyapunov exponent can be found from the relation $\lambda(x) = \int \log|f'(x)|\rho(x)dx$ where $\rho(x)$ is the invariant measure(note that if f is ergodic then $\rho(x)$ is unique).

In higher dimensions, for example in $\mathbb{R}^n$ the map f will have n Lyapunov exponents, say $\lambda_1, \lambda_2, ...\lambda_n$ for a system of n variables(in [19] it is explained a way how to calculate this exponents). Then the map is L-chaotic if the maximum Lyapunov exponent is positive i.e *max* $\{\lambda_1, \lambda_2, ...\lambda_n\}$ >0.

Proposition 2.3.9 Consider the continuous and differentiable map $f: \mathbb{R} \to \mathbb{R}$. If f has a positive Lyapunov exponent then f has also sensitive dependence on initial conditions.

Proof: Consider a point. $x_0 \in \mathbb{R}$. Now we chose a point $x_0^{'}$ close to x_0. Then from the relation (1) we have $\delta x_n = |f^n(x_0+\delta) - f^n(x_0)| = |x_n^{'} - x_n| = \delta x_0 e^{n\lambda(x_0)} = \delta$

where $\delta x_0 = |x_0^{'} - x_0|$ and $x_0^{'} = x_0 + \delta \Rightarrow e^{n\lambda(x_0)} = \dfrac{\delta}{\delta x_0} \Rightarrow n = \dfrac{1}{\lambda(x_0)} \log\left|\dfrac{\delta}{\delta x_0}\right|$

Choosing some δ for given x_0 then $\exists x_0^{'} \in N_\varepsilon(x)$ such that $\forall \varepsilon > 0$ after $m>n$ iterations we get

$|f^m(x_0^{'}) - f^m(x_0)| = \delta x_0 e^{m\lambda(x_0)} = \delta x_0 e^{(m-n)\lambda(x_0)}\, e^{n\lambda(x_0)} = e^{(m-n)\lambda(x_0)}\, \delta > \delta \Rightarrow f$ has sensitive dependence on initial conditions.

Proposition 2.3.10 Every expanding map $f:\mathbb{R} \to \mathbb{R}$ has sensitive dependence on initial conditions.

Proof: Since the map f is expanding from definition 2.1.8 we have that $|f'(x_i)| > 1$ $\forall x_i \in \mathbb{R}$. Now the Lyapunov exponent of f at the point x is given by the formula

$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log|f'(x_i)|$. Since $|f'(x_i)| > 1 \quad \forall x_i \in \mathbb{R} \Rightarrow$

$\log|f'(x_i)| > \log 1 = 0 \Rightarrow \sum_{i=0}^{n-1} \log|f'(x_i)| > 0 \Rightarrow$ Dividing both sides of the inequality with n and taking the limit for $n \to \infty$ we get $\lambda(x)$. From proposition 2.3.8 we have that f is sensitive.

2.4 Examples

In this section of my project I will examine the chaotic behaviour(with respect to the three definitions of chaos in the last section) of some interesting and well-known maps .

Example 2.4.1 Consider the Bernoulli shift map $B(x):[0,1) \to [0,1)$ given by

$$B(x) = 2x \bmod 1 = \begin{cases} 2x & 0 \le x < 0.5 \\ 2x-1 & 0.5 \le x \le 1 \end{cases}.$$

The graphs of $B(x)$ and $B^2(x)$ are shown in figures 1 and 2 respectively.

First I will prove that $B(x)$ is transitive using symbolic dynamics. We let Σ be the metric space of all infinite sequences containing 0's and 1's equipped with the metric $\rho(s,\tau)=\frac{1}{2^i}|s_i-\tau_i|$ $\forall$ $s=(s_0s_1s_2...)$ and $\tau=(\tau_0\tau_1\tau_2...)$ $\in$ Σ and we define $\sigma:\Sigma\to\Sigma$ given by $\sigma(s_0s_1s_2...)=(s_1s_2s_3...)$. Then there exist a point $x=(0100011011000001...)$ created by blocks of 0's and 1's,which has a dense orbit. So σ is transitive and then $B(x)$ is transitive. [24]

Now I will prove that $B(x)$ has a dense set of periodic points.

We have that Fix(B) = $Per_1(B)$ = $\{0\}$ $\Rightarrow|Per_1(f)|=1=2^1-1$. The second iterated map B^2 is given by $B^2(x)=4x\bmod 1$ and $Per_2(B)=\left\{0,\frac{1}{3},\frac{2}{3}\right\}$ $\Rightarrow$ $|Per_2(B)|=3=2^2-1$. Generalizing this result the n-th iterated map is given by $B^n(x)=2^n x\bmod 1$. So

$Per_n(B)=\left\{0,\frac{1}{2^n-1},\frac{2}{2^n-1},...,\frac{2^n-2}{2^n-1}\right\}$ and $|Per_n(B)|=2^n-1$

Now $\lim_{n\to\infty}|Per_n(B)|=\infty$ so $\forall x\in[0,1)$ and $\forall\varepsilon>0$, $N_\varepsilon(x)$ will contains a periodic point. Hence the periodic points of B are dense.

Also since $B(x)=2x\bmod 1$ then $B'(x)=2$ $\forall x\in[0,1)$ except for $x=0.5$ where the derivative is not defined $\Rightarrow$ $\lambda(x)=\log|B'(x)|=\log 2>0$ $\Rightarrow B(x)$ has a positive Lyapunov exponent.

So all the conditions for the three chaotic definitions are satisfied so the map B is D-chaotic, W-chaotic and L-chaotic.

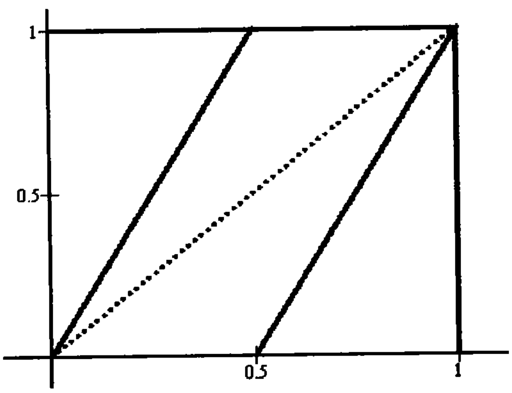


Figure 1 $B(x)$

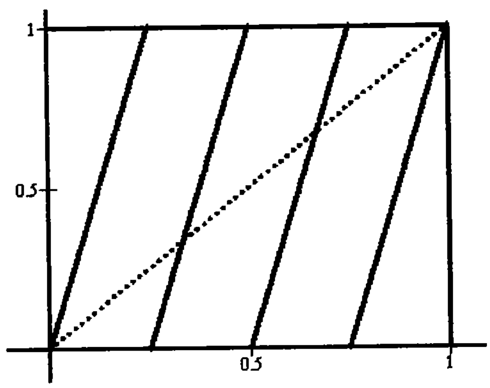


Figure 2 $B^2(x)$

Example 2.4.2 Consider the continuous map $f: X \to X$ defined by $f(e^{i\theta}) = e^{2i\theta}$ and

$X = S^1 \setminus \left\{ e^{\frac{2\pi pi}{q}} : p, q \in \mathbb{Z}, q \neq 0 \right\}$ is a metric space equipped with the arclength metric

d. Now every non empty subset of X is eventually expanded under iteration to cover *X,* so f is transitive. Also by defining in this way the set *X* we let out all the periodic points of f, so *f* has no (dense) periodic points. Finally for any given two points in *X,* say $e^{i\theta}$ and $e^{i\phi}$ such that $0 < |\theta - \phi| < \pi$ we can choose n that satisfies $2^n < |\theta - \phi| \leq \pi < 2^{n+1} |\theta - \phi|$ $\Rightarrow f$ is sensitive with sensitivity constant $\frac{\pi}{2}$ since $d(f^n(e^{i\theta}), f^n(e^{i\phi})) > \frac{\pi}{2}$. So the map *f* is L-chaotic, not D-chaotic and not W-chaotic [16].

Example 2.4.3 Consider the continuous map $f: Y \to Y$, where $Y = S \times [0,1]$ is a metric space equipped with the "taxicab" metric $d((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| - |y_2 - y_1|$ for every pair (x_1, y_1) and $(x_2, y_2) \in Y$. We define f by $f(e^{i\theta}, t) = (e^{2i\theta}, t)$.

Clearly a point $z = (e^{i\theta}, t)$ will be a periodic point for f when $e^{i\theta}$ is the root of unity of order $2^n - 1$ for some n .So the periodic points of f are dense in *Y.* On the other hand if we take two sets A and *B*, where $A = S^1 \times \left[0, \frac{1}{2}\right)$ and $B = S^1 \times \left(\frac{1}{2}, 1\right]$ then $\forall\ n \in \mathrm{N}$ we have that $f^n(A) \cap B = A \cap B = \varnothing \Rightarrow$ The map is not transitive. Finally if we work in the same way as in the last example it's easy to conclude that the map f is sensitive. So the map *f* is not D-chaotic and not W-chaotic [16].

Example 2.4.4 Consider the map $f: \mathbb{R} \to \mathbb{R}$ given by $f(x)=2x$. The graphs of $f(x)$ and $f^2(x)$ are shown in figures 3 and 4 respectively.

First I will check if the transitivity condition is satisfied. We can observe(from the graph) that for $x>0$ $f^n(x) \to \infty$ when $n \to \infty$. On the other hand for $x<0$ $f^n(x) \to -\infty$ when $n \to \infty$ so there does not exist an orbit that goes from $x<0$ to $x>0$ or vice versa and the map can be decomposed into two open disjoint sets. So the transitivity condition is not fulfilled and the map f is not D-chaotic, not W-chaotic and not L-chaotic.

It is interesting to check if the other conditions are satisfied. It's easy to verify that the map has only one fixed point at $x = 0$ so $Fix(f) = \{0\}$ and also this fixed point is the only periodic point with period n $\Rightarrow Per_n(f) = \{0\} \Rightarrow f$ has not a dense set of periodic points.

Since $|f'(x)| = 2 \ \forall \ x \in \mathbb{R} \Rightarrow \lambda(x) = log2 > 0 \ \forall \ x \in \mathbb{R}$. So f has a positive Lyapunov exponent and from proposition 2.3.9 is sensitive.

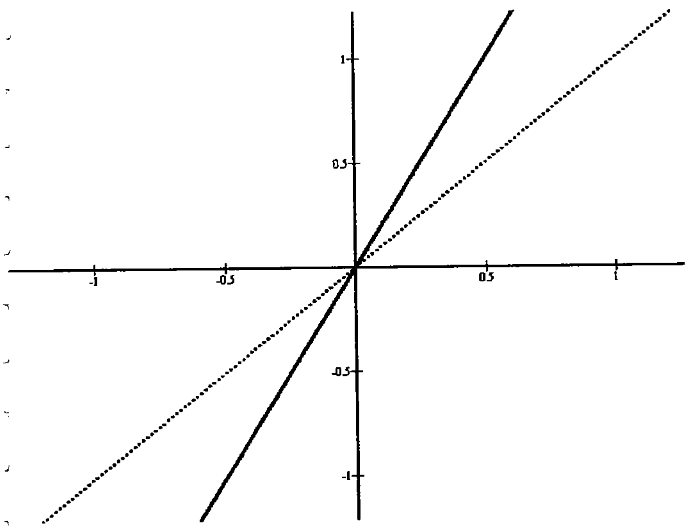


Figure 3 $f(x)$

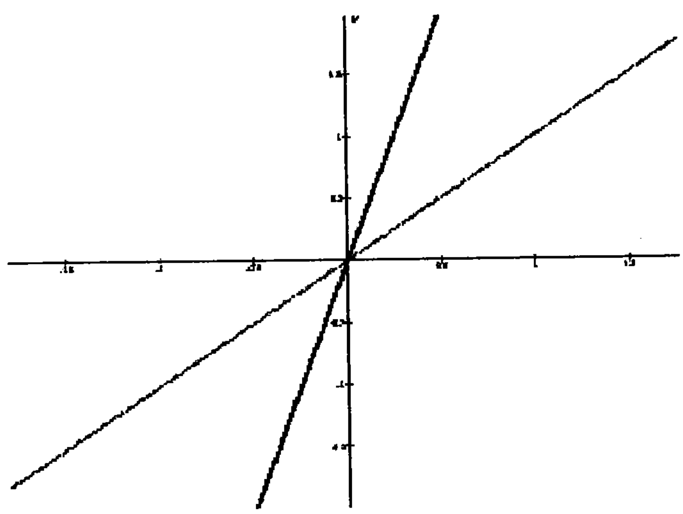


Figure 4 $f^2(x)$

Example 2.4.5 Consider the tent map $T: [0,1] \to [0,1]$ given by

$$T(x) = \begin{cases} 2x & 0 \le x \le 0.5 \\ 2-2x & 0.5 \le x \le 1 \end{cases}$$

The graphs of $T(x)$ and $T^2(x)$ are shown in figures 5 and 6 respectively.

First I will prove that $T(x)$ is transitive. So we choose a positive number d such that $0 < d < \frac{1}{2}$ and the compact interval $I = \left[\frac{1}{2}d, d\right]$. Then $\exists \ k \in \mathbb{N}$ such that $2^{k-1}d < \frac{1}{2} < 2^k d$. The k-th iteration of $T(x)$ gives $T^k(I) = \left[2^{k-1}d, 2^k d\right]$ and the k+1-

th iteration gives $T^{k+1}(I)=T\left(\left[2^{k-1}d,\frac{1}{2}\right]\cup\left(\frac{1}{2},2^k d\right]\right)=T\left(\left[2^{k-1}d,\frac{1}{2}\right]\right)\cup T\left(\frac{1}{2},2^k d\right]$

$=\left[2^k d,1\right]\cup\left[2-2^{k+1}d,1\right)$

Continuing in the same way the k+2-th iteration of T is $T^{k+2}(I)=\left[0,2(1-2^k d)\right]\cup T\left(\left[2-2^{k+1}d,1\right)\right)\Rightarrow 1-2^k d<\frac{1}{2}\Rightarrow\exists\quad m>0$ such that

$2^m(1-2^k d)>\frac{1}{2}\Rightarrow\left[0,\frac{1}{2}\right]\subset\left[0,2^m(1-2^k d)\right]\subset T^m\left(\left[0,2(1-2^k d)\right]\right)\subset T^{k+m+2}(I)$

So $\exists\quad k\in\mathbb{N}$ for every subinterval J of $[0,1]$ at which $T^k(I)\cap J\neq\varnothing\ \Rightarrow\ T$ is transitive and from proposition 2.3.4 T is also sensitive and has dense periodic points So T is D-chaotic , W-chaotic and L-chaotic. [26]

Now the odd extention of T in $[-1,1]$ is given by $K(x)=\begin{cases}-(2x-2) & -1\le x\le -0.5\\ 2x & |x|<0.5\\ 2-2x & 0.5\le x\le 1\end{cases}$

Then K is not transitive because the interval $(0,1)$ will never map onto any subinterval of $(-1,0)$. Also K is sensitive and has dense periodic points$\Rightarrow K$ is not D-chaotic, not W-chaotic and not L-chaotic.

But if we define a map $F(x)=-K(x)$ on $[-1,1]$ then F has dense periodic points since K has also. Basically every periodic point of K with period n will be a periodic point for F but with period $2n$. Also after some iterarions of F we can see clearly that F is transitive. Finally F is expanding so it is sensitive. So F is D-chaotic, W-chaotic and L-chaotic. [2]

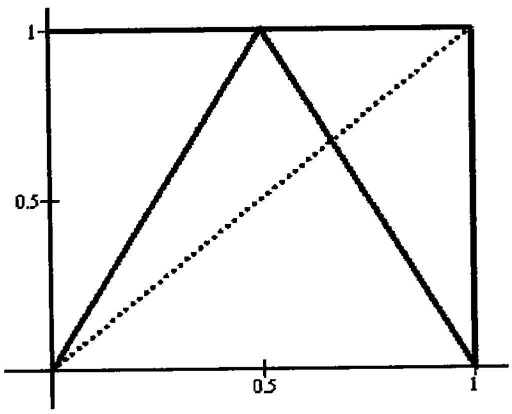


Figure 5 $T(x)$

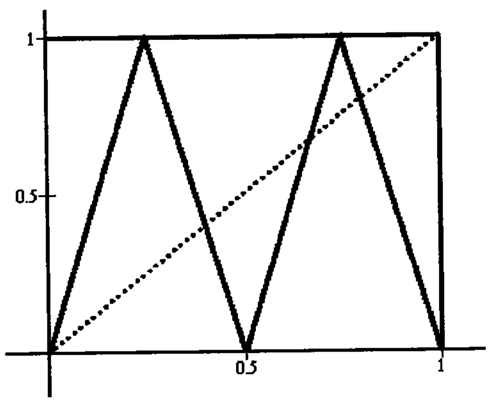


Figure 6 $T^2(x)$

Example 2.4.6 In [7] it is proved that the quadratic map F: $[0,1] \to [0,1]$ given by $F(x) = 4x(1-x)$ is D and W-chaotic. The graph of $F(x)$ is shown in figure 7. I will use this result to investigate the chaotic behaviour of the map $G(\rho,\theta) = (4\rho(1-\rho), \theta+1)$ using the polar coordinates (ρ,θ). The map G is defined on a disk $D(0,1) = \{x \in R^2 : \|x\| \leq 1\}$. After a finite number of iterations of the map G the image of a small disk in $D(0,1)$ will contain an open set $U \subset D(0,1)$ with a full radius. Also the rotation of *1* radian will spread U totally over $D(0,1)$ after a finite number of iterations. So G is transitive on $D(0,1)$.

Now since the quadratic map F is sensitive on $[0,1]$ then G is also sensitive.

Finally G has only a fixed point in the origin and does not have any periodic orbit of period $p>1$. Basically G shrinks or stretches the distance of every point of $D(0,1)$ from the origin while rotating by an angle of *1* radian. Since $1/\pi$ is irrational no point x_n that belongs to the orbit of x_0 can come back to the same ray which contains x_0. Hence G has no dense periodic points. So G is W-chaotic but not D-chaotic [6].

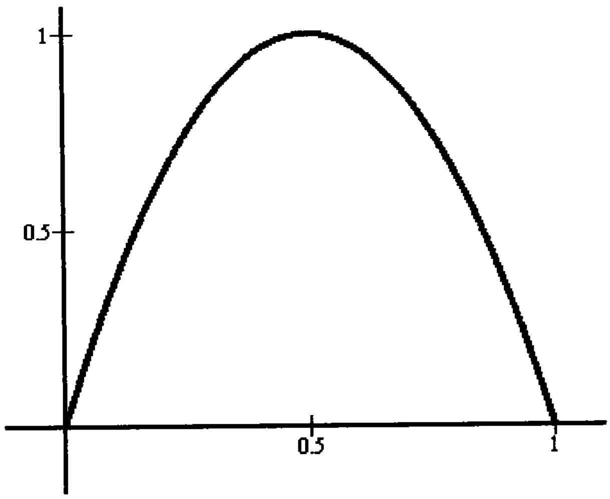


Figure 7 $F(x)$

Remark: In the last example since in the first coordinate the quadratic map F has dense periodic points and on the second coordinate the other map has not dense set of periodic points. From this result we can conclude that their "composition" together to the map G does not have dense periodic points.

3. Related definitions of chaos and crosslinks

3.1 Topological mixing and blending: An "alternative" of topological transitivity in Devaney's definition of chaos.

In this section I will explain two new concepts: Topological mixing and blending (strong and weak) and I will investigate if these two conditions can replace transitivity in Devaney's definition of chaos.

Definition 3.1.1 Consider the metric space X and the continuous map $f: X \to X$ .The map f is said to be topological mixing if for every pair of non-empty open sets U and V in X there exists a positive integer n such that $f^k(U) \cap V \neq \varnothing$ for every $k > n$.

Convention: Throughout this paper "mixing" will always mean "topological mixing".

Now I will give a condition for a map to be mixing on a compact interval. This result can be found in [4].

Lemma 3.1.2 Consider the compact interval I =[a,b]. Then there exist a continuous map $f: I \to I$ with $f(a)$=a and $f(b)$=b such that f is topological mixing.

Proof: First we choose numbers a_0, a_1, a_2, a_3 such that $a_0 = a$, $a_3 = b$ and $a_0 < a_1 < a_2 < a_3$.Also we choose the intervals $[a_0, a_1]$,$[a_1, a_2]$ and $[a_2, a_3]$ to have the same length. Now we let f to be the piecewise linear map such that $f(a_0) = a_0$, $f(a_1) = a_3$, $f(a_2) = a_0$ and $f(a_3) = a_3$.
So its clearly that for any subinterval J of an interval $[a_i, a_{i+1}]$,the length of $f(J)$ will be three times the length of J, hence for some n>0 we have that $f^n(J) = I$ and f is topological mixing.

Example 3.1.3 Let f be a continuous piecewise linear map on the compact interval $I = [0,1]$ such that $f(0) = 0, f(\frac{1}{2}) = 1$ and $f(1) = 0$. Then f is topological mixing because the assumptions of lemma 3.1.2 are satisfied. It can also be proved that f is a topologically transitive map (for example the Tent map satisfies the assumptions of this example).

From definition 3.1.1 clearly mixing implies transitivity. In [4] Block et al we have an equivalence between mixing and transitivity. Again I will give this result by a theorem but without its proof since it contains non trivial results. The proof also can be found in [4].

Theorem 3.1.4 Consider the compact interval X and the continuous map $f: X \to X$

Then the following are equivalent:

1. f is transitive and there exists a periodic point with odd period greater than one
2. f^2 is transitive
3. f is topological mixing

Remark: Not every transitive map is mixing.

Example 3.1.5 Let S^1 be the unit circle and let f: $S^1 \to S^1$ be an irrational rotation. Then the map is transitive but its not mixing. More generally no translation (and no circle rotation) are mixing from the fact that translations preserve the natural metric on the torus induced by the standard Euclidian metric on $\mathbb{R}^n$ and from the fact that isometric are not mixing (for this result see [22])

In [2] Cranell suggests blending as an alternative of transitivity hypothesis in Devaney's definition of chaos. Blending is also a purely topologic condition as transitivity but as we will see above they are not equivalent.

Definition 3.1.6 Consider the continuous map f: $X \to X$ on a metric space X. Then f is said to be a strongly blending map if for every non-empty open sets $U,V \subset X$ $\exists n > 0$ such that $f^n(U) \cap f^n(V)$ contains an open set.

Also f is said to be a weakly blending map if for every non-empty open sets $U, V \subset X$ $\exists n > 0$ such that $f^n(U) \cap f^n(V) \neq \varnothing$.

Now I will give two main theorems that give a relation between blending (strong and weak) with topological transitivity. The first theorem is valid for any subset of $\mathbb{R}^n$ and its proof will be given. On the other hand the second theorem is valid only for compact spaces in the real line. Its proof is omitted and can be found in [2] .

Theorem 3.1.7 Consider the continuous map $f : X \subset \mathbb{R}^n \to X$ and let f has dense periodic points. Then if f is strongly blending then f is transitive.

Proof : Consider two non empty open subsets $U, V \subset X$. Then $\exists n > 0$ such that $M \subset f^n(U) \cap f^n(V)$, where M is an open set in X. Now let $Y := f^{-n}(M) \cap V$. Then since f is a continuous map and Y is also open (as intersection of two open sets) we can choose a periodic point q in Y with period $m > n$. But then $f^n(q) \in M$ and there exist $y \in U$ such that $f^n(q) = f^n(y)$.

Finally we have $f^m(y) = f^{m-n}(f^n(q)) = f^m(q) = q$ and so $q \in f^m(U) \cap V \neq \varnothing$ and the theorem is proved.

Before giving the second theorem I will define the repelling fixed point.

Definition 3.1.8 Consider the continuous and differentiable map f: $X \to X$ and let $p \in X$ to be a fixed point for the map f. Then p is said to be a repelling fixed point for the map f if $|f'(p)| > 1$.

Theorem 3.1.9 Let f: $I \to I$ be a continuous map on the compact interval I. Then if f has a repelling fixed point and f is also transitive then f is weakly blending.

Remark: The converse of theorem 3.1.7 does not hold since there exist transitive maps without being strongly blending.

Example 3.1.10 Consider the map f: $S^1 \rightarrow S^1$ defined by $f(\theta) = \theta + k$, where $\frac{k}{\pi}$ is an irrational number. Then f is transitive but not strongly blending (also f is not weakly blending).

Remark : The converse of theorem 3.1.9 also is not true since there exist weakly blending maps without being transitive.

Example 3.1.11 The map F defined on example 2.4.5 as we have seen is not transitive but it is weakly blending since every open subinterval of $[-1,1]$ eventually maps onto another subinterval which contains the fixed point of F which is located at the origin.

3.2 Expansivity: an alternative of sensitivity on Wiggins' chaos

Expansivity is a condition related directly with sensitive dependence on initial conditions but these two conditions are clearly not equivalent as I will explain later on.

Definition 3.2.1 Consider the metric space X equipped with the metric d and the map $f: X \rightarrow X$. Then f is said to be expansive if there exists a positive number $c>0$ such that if $x,y \in X$ and $x \neq y$, then $\exists n > 0$ such that $d(f^n(x), f^n(y)) \geq c$.
This positive number c is called an expansive constant for f.

Remark: Clearly from the definition 3.2.1 expansivity implies sensitivity because in expansivity all nearby points of x separate by at least c (in sensitivity condition we need only one point to have this property). If a map is expansive then any two orbits become at least a fixed distance apart. On the other hand trivially sensitivity does not imply expansivity.

3.3 Some more crosslinks between chaotic ingredients

In this section I will try to explain how all these chaotic ingredients (transitivity, mixing, denseness, expanding maps, positive Lyapunov exponents e.t.c) are interrelated. Some of these relations I have already talked about are:

f is expanding $\rightarrow f$ has $\lambda(x) > 0$ $\rightarrow f$ is sensitive

mixing $\rightarrow$ transitivity

transitivity + dense periodic points $\rightarrow$ sensitivity

On the following I will prove that:

f is expanding $\rightarrow f$ is mixing $\rightarrow f$ is sensitive

Proposition 3.3.1 Consider the unit circle S^1. Then every expanding map $f : S^1 \rightarrow S^1$ is topologically mixing [22].

Proof: First we suppose that there exists a positive number k such that $|f'(x)| \geq k > 1$. $\forall x \in \mathbb{R}$. Consider now the map F: $\mathbb{R} \rightarrow \mathbb{R}$ such that $f \circ \pi = \pi \circ F$ (here π is the map $\pi : \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$ given by $\pi(x) = x \, mod 1$). Then clearly we have that $|F'(x)| \geq k$ $\forall x \in \mathbb{R}$. Now we consider the compact interval $[a,b]$. From the mean value theorem we know there exists $c \in [a,b]$ such that $|F'(c)| = \left| \frac{F(b) - F(a)}{b-a} \right|$ $\Rightarrow$ $|F(b) - F(a)| = |F'(c)| |b-a| \geq k |b-a|$. According to the last relation the length of any interval will be increased under F^n at least by a factor k^n so for every interval I there will exist a natural number n such that the length of $F(I)$ will be bigger than one. Then the image of the projection of the interval I to S^1 under f^n will contain S^1. Finally every open set has an image under an iterate of f that contains S^1 because every open set of S^1 contains an interval. Hence the theorem is proved.

In example 2.4.1 I studied the Bernoulli shift map $B(x)$ in the unit interval given by $B(x)=2x \bmod 1$. As we have seen this map can be represented on the unit circle. This map belongs to the family of linear expanding maps $E_m: S^1 \to S^1$ given by $E_m(x)=mx \bmod 1$. Before I give a corollary on the chaotic behaviour of these maps I will define the linear map.

Definition 3.3.2 Consider the map $f: X\to X$. Then f is said to be a linear map if $f(ax+by)=af(x)+bf(y)$ $\forall x,y\in X$ and $\forall a,b\in\mathbb{R}$.

Corollary 3.3.3 Linear expanding maps are D-chaotic, W-chaotic and L-chaotic.

Proof: From proposition 3.3.1 E_m is mixing and hence is transitive. On the other hand E_m has m^n-1 periodic points of period n (generalizing the result for the Bernoulli shift), i.e $|Per_n(E_m)|=m^n-1$ .So as $n\to\infty$ then $Per_n(E_m)\to\infty$,hence the periodic points of E_m are dense. From theorem 2.3.2 E_m is also sensitive and the theorem is proved.

Proposition 3.3.4 Consider the metric space X equipped with the metric d and the continuous map $f: X\to X$ .Then if f is a mixing map then it is sensitive.

Proof: Consider a positive number d and two points say $p,\ q\in X$ such that $d(p,q)>4d$. Consider also the balls $B_d(p)$ and $B_d(q)$ both with radius d and centres p and q respectively. Now we take a point $x\in X$ and we choose an open neighbourhood of x say $N_\varepsilon(x)$ for some $\varepsilon>0$.Now since f is a mixing map then $\exists n_1,n_2\in\mathbb{N}$ such that $f^n(N_\varepsilon(x))\cap B_d(p)\neq\varnothing$ $\forall n>n_1$ and $f^n(N_\varepsilon(x))\cap B_d(q)\neq\varnothing$ $\forall n>n_2$. If we choose $n>\max\{n_1,n_2\}$ then $\exists y_1,y_2\in N_\varepsilon(x)$ such that $f^n(y_1)\in B_d(p)$ and $f^n(y_2)\in B_d(q)$.

Then we have $d(f^n(y_1),f^n(y_2))\geq 2\delta$. Also from the triangular inequality we get $(f^n(y_2),f^n(x))\geq\delta$ or $d(f^n(y_1),f^n(x))\geq\delta$. So f is sensitive with sensitivity constant 4δ.

4. Further details and further definitions

4.1 Discontinuity and chaos

In section 2.4 I studied some examples of linear maps, such as the Bernoulli shift map. This map is discontinuous x =0.5 but it is chaotic. Here I will study chaotic behaviour with respect of discontinuity. But first of all I will give some preliminaries that will be used in this section. .

Definition 4.1.1 Consider the metric X, a nonempty subset $M \subset$ **X** and a point $x_0 \in$ *M*. Then a point $x_0 \in M$ is said to be an interior point of M if $\exists\ \delta > 0$ such that $B_\delta(x_0) \subset M$. The set M is said to be rare(or nowhere dense) in X if its closure $\overline{M}$ has no interior points.

Definition 4.1.2 A topologic space X is said to be a Baire space if the countable union of any collection of closed sets with empty interior has empty interior (equivalently the interior of every union of countably many nowhere dense sets is empty).

The Baire category theorem gives sufficient conditions for a topological space to be a Baire space.

4.1.3 Baire category theorem Every non-empty complete metric space is a Baire space. More generally, every topological space which is homeomorphic to an open subset of a complete metric space is a Baire space. In particular, every topologically complete space is a Baire space.

In proposition 4.1.4 I will show that for maps defined on a Baire separable metric space which has at most one point of discontinuity(for more than one point this is false) transitivity implies the existence of dense periodic orbits.

Proposition 4.1.4 Let $f: X \to X$ be a transitive map on a Baire seperable space X equipped with the metric d. Then if f has only one point of discontinuity ,say p then there exists a point x such that its orbit is dense in X.

Proof: If $\overline{\{f^n(p)\}_{n\in\mathbb{N}}} = X$ then we finished. So we suppose that $\{f^n(p)\}_{n\in\mathbb{N}}$ is not dense in X. First I will prove that $\{f^n(p)\}_{n\in\mathbb{N}}$ is rare and I suppose that this is false. So there exist open sets $U, V \subset X$ such that $V \cap \overline{\{f^n(p)\}_{n\in\mathbb{N}}} = \varnothing$ and $U \subset \overline{\{f^n(p)\}_{n\in\mathbb{N}}}$. Since f is transitive then $f^m(q) \in V$ for some natural number m and for some $q \in U$. Also f^m is continuous at the point q because $f^m(q) \neq f^n(p)$ for each n. But then there exists an open subset W of V such that $f^m(W) \subset V$. Then by choosing a natural number k such that $f^k(p) \in W$, then we have $f^{k+m}(p) \in V$,hence we have a contradiction. Now we choose a (countable) basis $\{A_n\}_{n\in\mathbb{N}}$ of open sets in $X \setminus \overline{\{f^n(p)\}_{n\in\mathbb{N}}}$ and we define $F_n = \{x \in X : f^m(x) \in A_n \text{ for some } m\}$. So we fix $n \in \mathbb{N}$ and then if $x \in F_n$ then $\exists\, m \in \mathbb{N}$ with $f^m(x) \in A_n$ and also f^m is continuous at the point x. As a consequence of this there exists an open neighbourhood U of x with the property $f^m(U) \subset A_n$ and hence F_n is open. Now for any open set $Y \subset X$ because of the transitivity of f we can find $m \in \mathbb{N}$ and $y \in Y$ such that $f^m(y) \in A_n$ .So $y \in F_n$ and F_n is dense . Finally we define $F := \bigcap_{n\in\mathbb{N}} F_n$ and because X is a Baire space then F is dense in X and thus $\overline{\{f^n(x)\}_{n\in\mathbb{N}}} = X \quad \forall\, x \in F$ and the proposition is proved.

Remark: This proposition does not hold if the map has more than one point of discontinuity. In example 2.4.6 I defined the Tent map $T(x)$. As we have seen it is a continuous map on the unit interval. Now we fix $y_1 \in (0,1)$ such that $\{T^n(y_1)\}_{n\in\mathbb{N}}$ is dense in $[0,1]$ and we let $y_0 = y_1 + 1$

We define f: $[0,2] \to [0,2]$ given by

$$F(x) = \begin{cases} T(x) & x \in [0,1] \\ y_0 & x=1 \\ 1+T(x-1) & x \in (1,2) \\ y_1 & x=2 \end{cases}$$

In [13] it can be found the proof that $F(x)$ does not have a dense orbit but it is transitive. I will only prove that F does not have a dense orbit.

Proof: First we have that $\{T^n(y_1)\}_{n\in\mathbb{N}} = \{f^n(y_1)\}_{n\in\mathbb{N}}$ is dense in (0,1). Now this sequence agrees with $\{T^n(y_0-1)\}_{n\in\mathbb{N}}$ so $\{x_n\}_{n\in\mathbb{N}} := \{T^n(y_0-1)\}_{n\in\mathbb{N}}$ is dense in (1,2). Also we have that $f(y_0) = 1+T(y_0-1)$ and $f(x_1) = 1+T(x_1-1) = 1+T^2(y_0-1) = x_2$. By induction we have $x_{n+1} = f(x_n)$ and hence $\{f^n(y_0)\}_{n\in\mathbb{N}}$ is dense in (1,2).

Now if $x = k/2^n$ $x = 2/k^n$, where $k,n \in \mathbb{N}$ then $\exists\ m \in \mathbb{N}$ with $f^m(x) = y_0$ when $x = 2/k^n$ and $f^m(x) = y_0$ when $x \in (0,1]$ and $f^m(x) = y_1$ when $x \in (0,1]$. From the map $T(x)$ we have $T(\frac{1}{2}) = 1, T(\frac{1}{4}) = 1, \ldots T^n(\frac{k}{2^n}) = 1\ \forall k/2^n \in (0,1)$. So for $x = k/2^n \in (0,1)$ $\exists\ m \in \mathbb{N}$ with $f^m(x) = f(1) = y_0$.

For $x \in (1,2)$ and $f(x) > 1$ then either $f(x) = 2$ we get $f^2(x) = y_1$ or we have $f(x) \in (1,2)$. So for $f(x) \in (1,2)$ we iterate the map until we find $m \in \mathbb{N}$ such that $T^m(x-1) = 1$. Since $f^\lambda(x) = 1+T^\lambda(x-1) = f(1) = y_0$ where $1 \le \lambda \le m$ then we get $f^{m+1}(x) = y_1$. So the map f has not a dense orbit.

The proof that f is transitive is omitted and can be found also in [13].

Remark: In [14] it is proved for a map that is one to one in an interval, chaotic behaviour cannot occur when f is continuous but can occur if f has even one discontinuity (of course there are some maps that are not one to one and exhibits chaotic behaviour).

4.2 Li-Yorke's chaos and scrambled sets

In this section I will try to explain chaos in a different view than in section 2.4. Here I will explain the meaning of the scrambled sets and I will define Li-Yorke's chaos.

Definition 4.2.1 Consider an interval I and the continuous map f: $I \to I$. Then an uncountable subset S of I containing no periodic points of f is said to be scrambled if:

1. Any periodic point p of f and any point $x \in I$ satisfies $\limsup_{n\to\infty} |f^n(x) - f^n(p)| > 0$.
2. $\forall\, x, y \in X$ $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$ and $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$.

Clearly condition 1 requires that for an orbit starting from a point in S does not approach asymptotically any periodic orbit (Generally two orbits $\{x, f(x), f^2(x)\}$ and $\{y, f(y), f^2(y)\}$ approach asymptotically if $\lim_{n\to\infty} |f^n(x) - f^n(y)| = 0$). The second condition requires that two arbitrary orbits starting from two different points in S can be close to each other but cannot approach each other asymptotically. Both conditions are certainly related with sensitivity of f.

4.2.2 Li-Yorke's definition of chaos Let I be an interval and let f: $I \to I$ be a continuous map with a periodic point of period three. Then f is said to be chaotic in the sense of Li-Yorke or L-Y chaotic if f has an uncountable scrambled set.

In the original paper [7] Li and Yorke prove the following theorem (The proof can be found also in the same paper):

Theorem 4.2.3 Let I be an interval and let $f: I \to I$ be a continuous map. If there exists a point $x \in I$ such that $y = f(x)$, $z = f^2(x)$ and $w = f^3(x)$ satisfying $w \le x < y < z$ (or $w \ge x > y > z$) then

1. $\forall\, n = 1,2,3,\ldots..$ there exists a periodic point in I with period n.
2. There exists an uncountable scrambled set $S \subset I$ with no periodic points.

Note: The hypothesis of theorem 3.5.3 is also satisfied if f has a periodic point with period three.

Li –Yorke's chaos has two important disadvantages in contrast with the other three definitions on section 2.4. The first is that it can only be used on intervals on the real line and not in higher dimensional spaces. For example the rotation on $\mathbb{R}^2$ of 120° has a periodic point with period three but it does not have an uncountable scrambled set. The second disadvantage is that it cannot be applied to maps even with one discontinuity since discontinuity is critical to L -Y chaos.

Now I will examine the chaotic behaviour of same maps with respect to Li –Yorke chaos.

Example 4.2.4 Consider the map f: [-1,1] $\rightarrow$ [-1,1] given by $f(x)=2|x|-1$. The figure of $f(x)$ is shown in figure 8.Now we can observe that $f(\frac{-7}{9})=\frac{5}{9}, f(\frac{5}{9})=\frac{1}{9}$ and $f(\frac{1}{9})=-\frac{7}{9}$ so f has a periodic point of period 3 and f is L-Y chaotic [27].

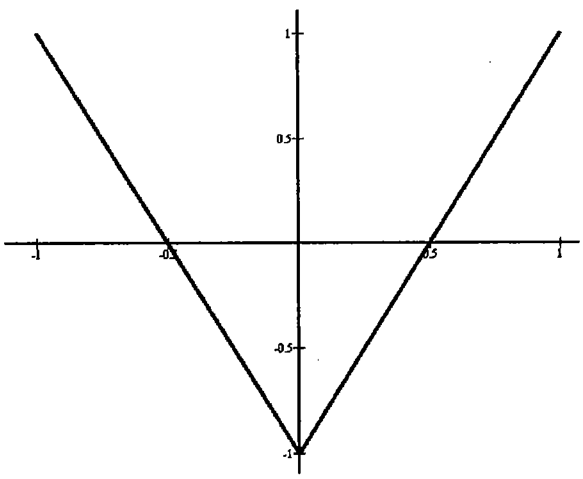


Figure 8 $f(x)$

Example 4.2.5 Consider the map f: [0,1] $\rightarrow$[0,1] given by

$$f(x)=\begin{cases} x+1/3 & 0\le x\le 2/3 \\ 1-7(x-2/3) & 2/3\le x\le 2/3+1/8 \\ x-2/3 & 2/3+1/8\le x\le 1 \end{cases}$$

The figure of $f(x)$ is shown in figure 9. Choosing the interval I=[1/8,1/3] we can see that it is transported to the interval J=[1/3+1/8,2/3]. Continuing in the same way J is transported to K=[2/3+1/8,1] and finally K to I. So because of this all points in these intervals are period three points, hence the hypothesis of definition 3.5.2 is satisfied and there exists an uncountable scrambled set, hence exhibits Li –Yorke chaos.

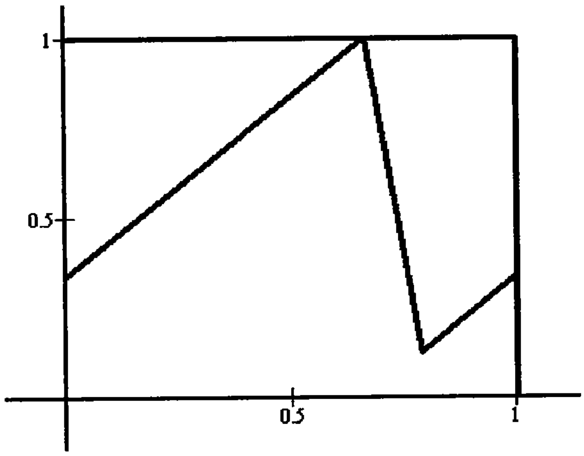


Figure 9 $f(x)$

Example 4.2.6 Consider the map $f_a:[0,1)\to[0,1)$ given by $f_a(x)=frac(x-a)$, where $frac(x)$ denotes the fractional part of x and a is an irrational number. First we choose two arbitrary points x and y in [0,1) such that $x<y$. Then we have

$$\left|f_a^n(x)-f_a^n(y)\right|=\left|f_{na}(x)-f_{na}(y)\right|=\begin{cases}1-|x-y| & \text{x<frac(na)}\\ |x-y| & \text{otherwise}\end{cases}$$

So from these we have $\liminf_{n\to\infty}\left|f_a^n(x)-f_a^n(y)\right|=\min\{|x-y|,1-|x-y|\}>0$ and from these there cannot exist an uncountable scrambled set so the map is not L-Y chaotic.

Example 4.2.7 Another map that is L-Y chaotic is the generalised logistic map $f(x)=ax(1-x/b)$ with $a\in(3.84,4)$ and $f(x)=\max\{ax(1-x/b),0\}$ for $a>4$ both defined in the interval $I=[0,K]$. The analysis of this example can be found in [7].

Remark : In [11] it is proved that for a continuous and transitive map $f: X\to X$, where X is a compact metric space(in the real line) containing a fixed point or a periodic point there exists an uncountable scrambled set, and hence Devaney's chaos implies Li-Yorke's chaos.

4.3 Topological and metric entropy: A chaos criterion

In this section I will give a slightly different sense of chaos explaining the topological and metric entropy with respect to chaos. As I will explain later topological entropy is related to Li-Yorke's chaos and metric entropy with the Lyapunov's chaos. In particular a Dynamical System is said to be chaotic if it has a positive topological or metric entropy. The determination of these two entropies is very difficult and that's why we prefer to calculate the Lyapunov exponent of the system to study its chaotic behaviour.

4.3.1 Topological entropy

I will define the topological entropy for a continuous map $f: I \to I$ where $I=[a,b]$ is a compact interval of the real line. So let α and β be two open covers for the compact interval I . Then their join $\alpha \vee \beta$ will be also an open cover for I(see [25] for the proof) consisting of all sets $A \cup B$ with $A \in \alpha$ and $B \in \beta$. Also when the inverse of f exists and it is continuous, $f^{-1}\alpha$ is an open cover for I consisting of all the sets $f^{-1}\alpha$ where $A \in \alpha$(again see [25] for the proof).

So if α is an open cover for I, since I is compact there exists an open subcover γ of α for the interval I. The entropy $H(\alpha)$ of an open cover α is defined as $H(\alpha)=\log N(\alpha) \geq 0$ where $N(\alpha)$ is the minimum number of open sets in any finite subcover. Clearly $H(\alpha)=0$ when $I \in \alpha$.

Definition 4.3.2 Consider the continuous map $f: I \to I$ where is a compact interval on the real line. Let $h(f,\alpha)=\lim_{n\to\infty} H(\alpha \vee f^{-1}\alpha \ldots \vee f^{-n+1}\alpha)/n$ be the entropy of f related to the cover α.We define the topological entropy of the map f as $h(f)=\sup_{\alpha} h(f,\alpha)$ where the supremum is taken over all covers α.

Clearly we can see that $0 \leq h(f,\alpha)) \leq H(\alpha)$ and $0 \leq h(f) \leq \infty$.

Remark :

1) The number $H(\alpha \vee f^{-1}\alpha \vee \ldots \vee f^{-n+1}\alpha)$ is the smallest number of open sets in I that is necessary to cover I, i.e is the most efficient open cover of I.
2) It can be proved that when f is a homeomorphism then $h(f) = h(f^{-1})$ (see [4]).
3) If we consider the map f_a as defined in example 3.5.6 where a is an irrational number then f_a is discontinuous at a so we cannot define the topological entropy. But if f_a be taken as rotation in the circle group then any such rotation has $h(f)=0$ [14].
4) A map can have $h(f)>0$ but can have no periodic points. This can be indicated with the following counter example:

Example 4.3.3 First we choose a map with $h(f)>0$ and we consider the direct product of this map with an irrational rotation R_a (a is the irrational number). Then we use the following formula (it can be found in [22] together with its proof) $h_{top}(f \times R_a) = h_{top}(f) + h_{top}(R_a)$ and we get $h_{top}(f \times R_a) = h_{top}(f) > 0$.

The map $f \times R_a$ has positive topological entropy but has no periodic points.

4.3.4 Topological entropy for piecewise maps

In the case when the map f is piecewise monotonic over the interval I then the topological entropy of f is given by $h(f) = \lim_{n\to\infty} \log lap(f^n)/n$, where $lap(f^n)$ is the lap number of the iterated map f^n, i.e the number of monotonic pieces of f^n on the interval I.

For example if we consider the tent map $T(x)$ on the unit interval then lap(T)=2 since $T(x)$ has two monotonic pieces in [0,1]. Also for the second iterated Tent map we have lap(T^2)= 4.

4.3.5 Topological entropy and chaos

Proposition 4.3.6 Consider the compact interval I and the continuous map $f: I \to I$. Then f is said to be chaotic if and only if $h(f)>0$

As I said before topological entropy is related to Li-Yorke chaos. According to [25] for a continuous map on a compact interval positivity of topological entropy implies Li-Yorke's chaos. The converse is clearly not true according again to [25].

Remark: According to [24] for a continuous map f defined on an interval I f has a positive topological entropy if and only if there exists a closed invariant set $J \subset I$ such that $f \mid J$ is chaotic in the sense of Devaney.

In the same paper is discussed the relation between transitivity and positivity of topological entropy. So in [24] the case that transitivity implies positive topological entropy is challenging since there exist metric spaces which a transitive map can have zero topological entropy. On the other hand the problem if positivity of topological entropy implies transitivity does not have a very good sense. We know that a map having two invariants sets I, J cannot be transitive but positivity of topological entropy can be caused by the fact that $f \mid I$ has positive entropy ($h(f) \geq h(f \mid I)$).

4.3.7 Metric-Kolmogorov entropy(K-entropy)

The metric (Kolmogorov) entropy is one of the most important measures by which chaotic motions can be characterised but it is very difficult to calculate its value . I will denote this metric entropy as $h(\mu)$. I will define the K-entropy for a continuous map f: $I \to I$ on a compact interval I.A Dynamical System that has a positive K-entropy is said to be a K-system [20].

4.3.8 Derivation of K-entropy

The definition of K-entropy is based on Shannon's formulation of degree of uncertainty in being able to predict the outcome of a probabilistic event. So we consider an experiment with r possible outcomes with the probabilities of each outcome be $p_1, p_2, ..., p_r$.

First of all we define the Shannon's entropy S to be a number that characterizes the amount of uncertainty that we have concerning which outcome will result and is given by $S = -\sum_{i=1}^{r} p_i \ln(p_i)$ .We define $p\ln(p)=0$ when $p=0$.

Now I will define the K-entropy. To do this we let I to be a compact interval containing the probability measure μ which is invariant under the map f. Now we partitioning the interval I into r disjoint subintervals, say $I_1, I_2, ..., I_r$ such that $I = I_1 \cup I_2 \cup ... \cup I_r$.

We define the entropy function $H(\{I_i\})$ for the partition $\{I_i\}$ to be $H(\{I_i\}) = \sum_{i=1}^{r} \mu(I_i) \ln[\mu(I_i)]^{-1}$. Then we construct a succession of partitions $\{I_i^{(n)}\}$ of smaller and smaller size. Let now $h(\mu, \{I_i\}) = \lim_{n\to\infty} \frac{1}{n} H(\{I_i\}^n)$, so we define the K-entropy to be the number $h(\mu) = \sup_{\{I_i\}} h(\mu, \{I_i\})$ where the supremum is taken over all possible initial partitions $\{I_i\}$. [20]

Example 4.3.9 We consider the map $f:[0,1] \to [0,1]$ defined by $f(x) = x + ax^z \, mod 1$, where $z \geq 1$. In [29] it is proved that $h(\mu) = 0$ for $z \geq 2$ and $h(\mu) > 0$ for $z < 2$ but the analysis of this example contains non trivial results.

Remark In [23] we have a relation that connects the topological and metric entropy of a map f which is $h(\mu) \leq h(f)$ with respect to the probability measure μ .

4.3.10 Chaos and connection to Lyapunov exponent

Theorem 4.3.11 Consider the compact interval I and the continuous map f: $I \to I$. Then the map f is said to be chaotic if f has a positive K-entropy.

The K-entropy for a continuous map is related to the Lyapunov exponent. For one dimensional maps the K-entropy is equal to the Lyapunov exponent.

In higher dimensions, say in $\mathbb{R}^n$ the map will have n Lyapunov exponents say $\lambda_1, \lambda_2, ... \lambda_n$ and the K-entropy is given by the averaged sum of positive Lyapunov exponents in the following integral form:

$h(\mu) = \int d^n x \rho(\vec{x}) \sum_i \lambda_i^+(\vec{x})$ where $\rho(\vec{x})$ is the invariant density and $\lambda_i^+(\vec{x})$ the positive Lyapunov exponents.

According to [20] Pesin (1976) proved that the metric entropy is given by $h(\mu) = \sum_{h_i > 0} h_i$, where h_i are the Lyapunov exponents of the map f with respect to the measure μ.

4.4 Chaotic ingredients under topological conjugation

In this section I will study the behaviour of some chaotic ingredients under topological conjugation. As I mentioned in section 4 transitivity and denseness of periodic points are topological properties and sensitivity is a metric property except if the space is compact. So clearly transitivity and denseness of periodic points are preserved under topological conjugation.

I will give now a counter example that proves that sensitivity (respectively expansivity) is not preserved under topological conjugation.

Example 4.4.1 Consider the spaces $X = (1, \infty) \subset \mathbb{R}$ and $Y = (0, \infty) \subset \mathbb{R}$, both equipped with the standard metric of the real line $d = |x - y|$. We let $f : X \to Y$ given by $f(x)=2x$ and $h: X \to Y$ given by $h(x)=\log x$. So(considering the diagram in definition 2.1.7) the map f is clearly sensitive but the map g is not sensitive since g is just a translation.

In this example the metric spaces X and Y are not compact. In the case where they are compact sensitivity and expansivity are preserved under topological conjugation. In the following theorem I will prove this result for a compact space in the real line for sensitivityy. For expansivity I will not prove this result since the proof is almost the same.

Theorem 4.4.2 Consider the compact metric spaces X and Y on the real line and the maps $f: X \to X$ and $g: Y \to Y$ such that f is conjugate to g. Then if g is sensitive (expansive) on Y then f is sensitive(expansive) on X.

Proof: Let $h: X \to Y$ be the topological conjugacy for the two maps and let $r>0$ be the sensitivity constant of g. I will denote the metric of the space X by d_X and the metric of Y by d_Y. Since the spaces are compact then h is clearly uniformly continuous so for the same $r>0$, $\exists\,\delta>0$ such that if $d_X(p,q)<\delta$ then $d_Y(h(p),h(q))< r$. So if $d_Y(h(p),h(q))\geq r$ then $d_X(p,q)\geq\delta$,or equivalently if $d_Y(p,q)\ \geq r$ then $d_X(h^{-1}(p),h^{-1}(q))\geq\delta$. Now if we choose a point $x\in X$ and take an $\varepsilon>0$ then $\exists\varepsilon_1>0$ so that if $q\in Y$ is within ε_1 of $y = h(x)$ then $p=h^{-1}(q)$ is within ε of x. Since g is sensitive at y then we take a $k\geq 0$ and also we take a $q\in Y$ that is within ε_1 of y. So if $p=h^{-1}(q)$ then we have $d_Y((g^k(y),g^k(q))\geq r$ and $d_X(h^{-1}(g^k(y)),h^{-1}(g^k(q))\geq\delta$. Also since $h^{-1}(g^k(y)=f^k(h^{-1}(y))=f^k(x)$ and $h^{-1}(g^k(q)=f^k(h^{-1}(q))=f^k(q)$ hence p is within ε of x and $d_X(f^k(x)f^k(q))\geq\delta$ and the proof is finished.

Now it is left to check if topological entropy of a map f is preserved under topologic conjugation. In [22] it is proved that topologic entropy is preserved under conjugation for a general metric space. I will give a more specified result about compact metric spaces on the real line. This result can be found in [4].

Proposition 4.4.3 Consider the compact metric spaces X and X and the maps the continuous maps $f:X\to X$ and $g:Y\to Y$. If there exists a homeomorphism θ:X→Y such that $\theta\circ f(x)=g\circ\theta(x)\quad\forall x\in X$ then $h(f)=h(g)$ i.e topological entropy is preserved under conjugacy.

Proof: Let $h\circ\theta=g\circ\theta$. Then by induction we obtain $\theta\circ f^k=g^k\circ\theta\quad\forall\ k>0$. So if we consider an open cover α of Y then $\theta^{-1}\alpha$ is an open cover of X. The topological entropy of g related to the cover α will be $h(g,\alpha)$= $\lim_{n\to\infty} H(\alpha \vee g^{-1}\alpha \vee ... \vee g^{-n+1}\alpha)/n = \lim_{n\to\infty} H(\theta^{-1}(\alpha \vee g^{-1}\alpha \vee ... \vee g^{-n+1}\alpha)/n$

$= \lim_{n\to\infty} H(\theta^{-1}\alpha \vee \theta^{-1}g^{-1}\alpha \vee ... \vee \theta^{-1}g^{-n+1}\alpha)/n$

$= \lim_{n\to\infty} H(\theta^{-1}\alpha \vee f^{-1}\theta^{-1}\alpha \vee ... \vee f^{-n+1}\theta^{-1}\alpha)/n = h(f,\theta^{-1}\alpha) \Rightarrow$ So we have that $h(g) \leq h(f)$

On the other hand since θ is a homeomorphism we have that $\theta^{-1} \circ g = f \circ \theta^{-1}$ and hence $h(f) \leq h(g)$. So $h(f) = h(g)$ and the proposition is proved.

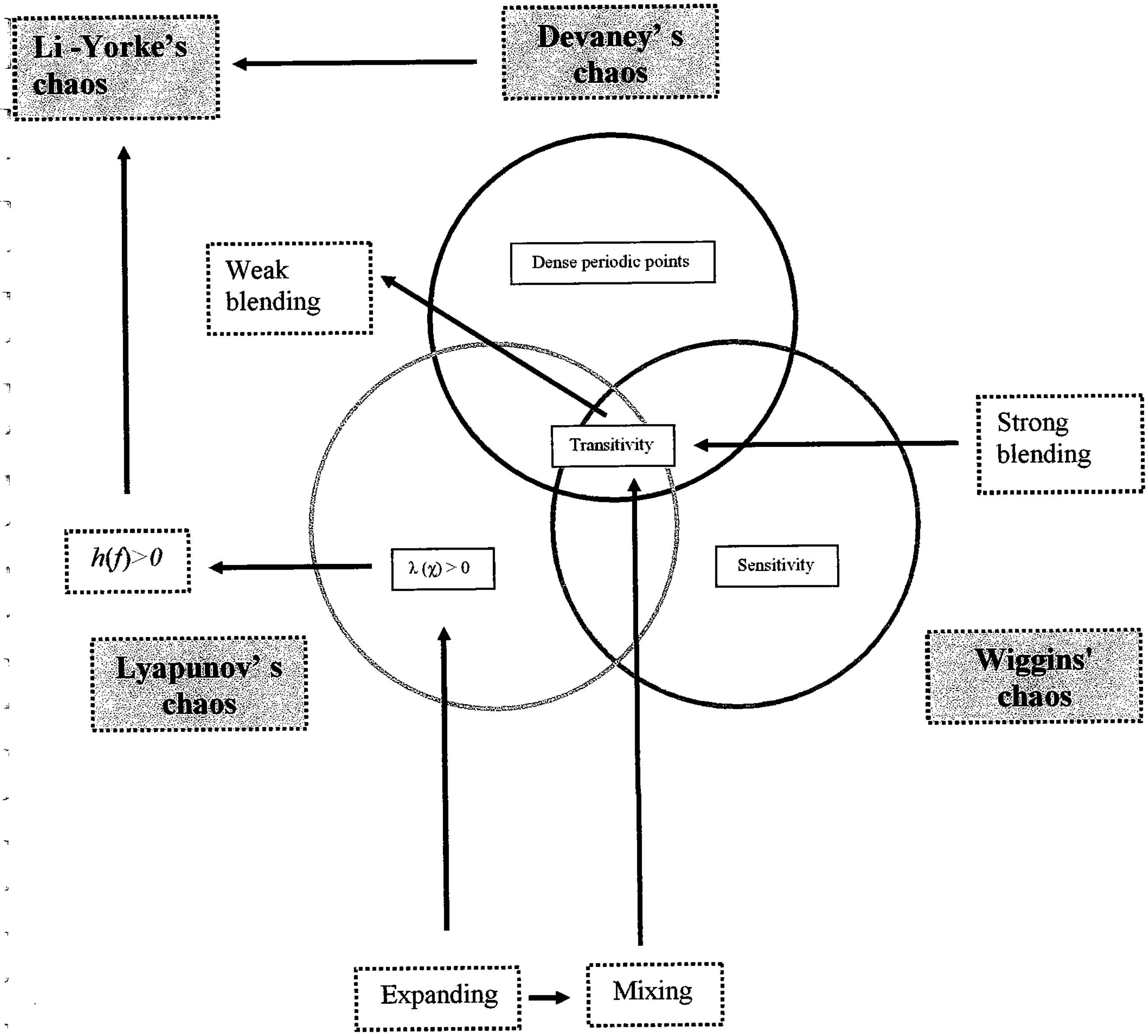
Remark:

1) It can be proved that the Lyapunov exponent is preserved under topological conjugation.
2) The metric entropy of a map is preserved under isomorphism (a bijective isometry between two spaces).

4.5 Overview and open questions

In this section I will give an overview of what I have already talked about and I will give some open questions that are still uncovered in this project. Finally I will give my own definition of chaos and I will explain the reason why I choose this definition.

In the following diagram I revise my project:


Li -Yorke's chaos
Devaney' s chaos
Weak blending
Dense periodic points
Transitivity
Strong blending
$h(f)>0$
$\lambda(\chi)>0$
Sensitivity
Lyapunov' s chaos
Wiggins' chaos
Expanding
Mixing

Although I obtain these interesting results some questions are still not answered. The unknown answers are the following:

W-chaos $\Rightarrow$ L-chaos $\Leftrightarrow$ mixing

mixing $\Rightarrow$ expanding

mixing $\Leftrightarrow$ D-chaos

Finally I will give my definition of chaos which is the following:

A continuous map $f: X \rightarrow X$ the metric space X is said to be chaotic if:

1. f is topologically transitive.
2. f exhibits sensitive dependence on initial conditions.

I prefer this definition of chaos since the sensitivity of the map can be checked very easily numerically and also I believe that transitivity is an essential ingredient of chaos. Furthermore an advantage of this definition of chaos is that it can be studied in dimensions higher than one.

5 Conclusion

This project provides an overview around the existing definitions of chaos. It introduces chaos in the sense of Devaney, Wiggins, Lyapunov and Li and Yorke. Also chaos is studied in terms of topological and metric entropy. Devaney's chaos is easy to be checked but contains the redundant hypothesis of sensitivity as I have already mentioned. As well Li-Yorke's chaos can be checked easily but it can only be used for maps on the real line and furthermore discontinuity (even and in one point) is critical. On the other hand Wiggins' chaos and Lyapunov's chaos can be checked more easily since sensitivity and the Lyapunov exponent of the map can be calculated numerically with an easy way. Nevertheless this advantage of these two definitions of chaos are not suitable since a map can be W-chaotic or L-chaotic without being D-chaotic. Additionally Devaney's chaos implies Wiggins' chaos , Lyapunov's chaos and Li-Yorke's chaos.

Finally the positivity of topological and metric entropy can be used as a chaos criterion but both these two entropies are very complicated to calculate numerically and it is preferred to determinine the Lyapunov exponent of the map to study chaos which is much more easier to be calculated.
Working on this project gave me the prospect to learn how to read and search specialised articles and books on Dynamical Systems although the results I obtained were not as interesting as could have been expected.
I wish to thank my supervisor Dr R. Klages for his advice and his unlimited help for writing and advising me in this project and also for introducing me to Chaotic Dynamical Systems.

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