Anomalous diffusion in random dynamical systems: Supplemental Material

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I. CALCULATION OF THE LYAPUNOV EXPONENT FOR RANDOM MAPS

For one-dimensional maps $M(x)$ obeying the deterministic equation of motion $x_{t+1} = M(x_t)$ the *local Lyapunov exponent* is defined by [1–6]

$$\lambda(x) = \lim_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t-1} \ln |M'(x_k)| \quad , \tag{1}$$

where $M'(x)$ denotes the derivative of the map. By definition here the Lyapunov exponent λ depends on the initial condition $x = x_0$ of the map, hence it is called local. Eq. (1) represents a time average along the trajectory $\{x_0, x_1, \ldots, x_k, \ldots, x_{t-1}\}$ of the map. If the map is ergodic the dependence on initial conditions will disappear [4–6]. For the piecewise linear map $M_a(x)$ defined by Eq. (1) in the main text we have that $M_a'(x) = a$, hence $\lambda(a) = \ln a$ as noted in the main text on p.1.

We now apply Eq. (1) to calculate the Lyapunov exponent for our random maps $R = M_a(x)$ as defined on p.1 and 2 in the main text, where $M_a(x)$ is again the piecewise linear map given by Eq. (1) in the main text. Therein the slope a becomes a random variable drawn from a probability distribution $\chi(\xi)$, $a = \xi$, at any time step t, $a = a_t$. We first choose for χ dichotomic noise, where a is sampled independently and identically distributed with probability $p \in [0, 1]$ from $a = a_{loc}$ and with probability $1 - p$ from $a = a_{exp}$. Feeding this information into Eq. (1) we obtain

$$\begin{aligned} \lambda_{dich} &= \lim_{t\to\infty} \left(\frac{t_{loc}}{t} \ln a_{loc} + \frac{t_{exp}}{t} \ln a_{exp} \right) \\ &= p \ln a_{loc} + (1-p) \ln a_{exp} \quad , \end{aligned} \tag{2}$$

where t_{loc} and t_{exp} yield respective numbers of events out of a total of t time steps for which the particle experiences the contracting and thus localising, respectively the expanding map [4]. In the long time limit these fractions are identical with the corresponding sampling probabilities p and $1 - p$ leading to our final result. Note that in our piecewise linear map $R = M_a(x)$ all particles are exposed to the same uniform slope a irrespective of initial conditions, hence $\lambda_{dich} = \lambda_{dich}(x)$.

From this equation we now calculate the critical sampling probability p_c at which $\lambda_{dich} = 0$ by choosing $a_{loc} = 1/2$ and $a_{exp} = 4$ as in the main text, which defines our first model. It follows

$$0 = p \ln 1/2 + (1-p) \ln 4 \quad , \tag{3}$$

which is solved to $p_c = 2/3$ as stated on p.2 in the main text.

By observing that dichotomic noise is defined by the probability distribution

$$\chi_{dich}(\xi) = p\delta(\xi - a_{loc}) + (1-p)\delta(\xi - a_{exp}) \tag{4}$$

we can rewrite Eq. (2) more generally as

$$\lambda_{dich} = \int_{a_{loc}-\epsilon}^{a_{exp}+\epsilon} d\xi \, \chi_{dich}(\xi) \ln |\xi| \tag{5}$$

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with $0 < \epsilon \ll 1$. From the above line of arguments it follows that for an arbitrary noise distribution $\chi(\xi)$, $\xi \in I$ we can calculate the Lyapunov exponent λ_χ of a random dynamical system $x_{t+1} = R(\xi_t, x_t)$ with map

$$R(\xi, x) = \begin{cases} \xi x & 0 \leq x < \frac{1}{2} \\ \xi(x-1)+1 & \frac{1}{2} \leq x < 1 \end{cases} \tag{6}$$

lifted onto the whole real line according to $R(x+1) = R(x) + 1$ by

$$\lambda_\chi = \int_I d\xi \, \chi(\xi) \ln |\xi| \quad . \tag{7}$$

Equations (5),(7) thus express the Lyapunov exponent of a random map R by ensemble averages over the corresponding noise distributions instead of time averages along trajectories, which more generally presupposes ergodicity of the dynamics [4–6]. For maps that are more complicated than our piecewise linear case $M_a(x)$ Eq. (1) in the main text, calculating λ_χ will require integration over the full invariant density $\rho(x, \eta)$ of the random map, which can be a very non-trivial object [7].

II. RANDOM MAP WITH UNIFORM DISTRIBUTION OF SLOPES

We now consider a second model that is more general than the one defined by dichotomic noise. For this purpose we sample the slopes of the piecewise linear map Eq. (6) from noise $\chi(\xi)$ uniformly distributed over an interval $\xi \in I = [a, b]$, $a > 0$, that is, we choose

$$\xi \sim \chi_{unif}(\xi) = \frac{1}{b-a} \quad . \tag{8}$$

For this model the Lyapunov exponent can be calculated from Eq. (7) to

$$\lambda_{unif} = \int_a^b d\xi \, \frac{1}{b-a} \ln \xi = b \log b - b - (a \log a - a) \quad . \tag{9}$$

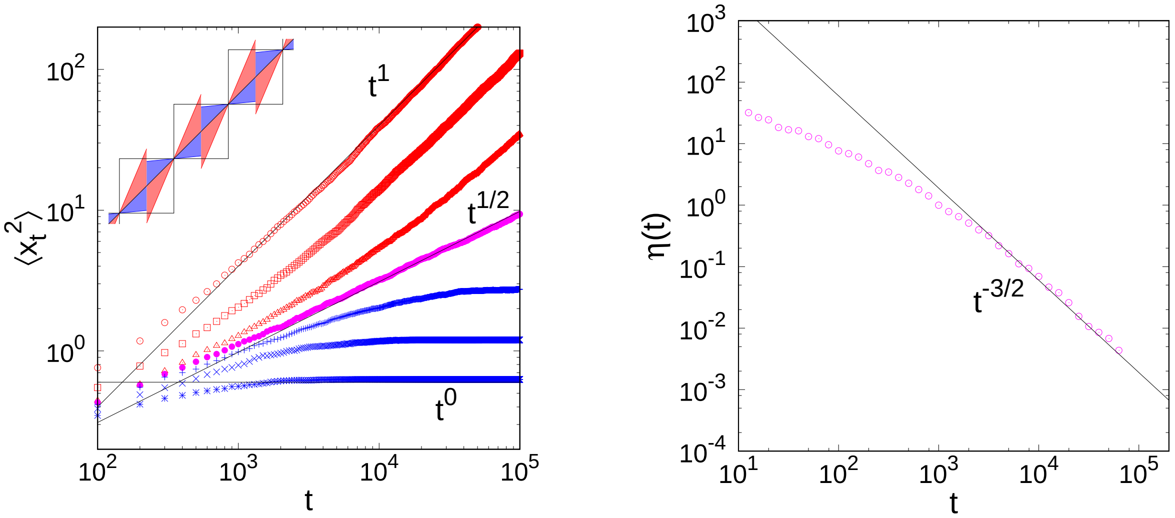

FIG. 1: Mean square displacement (MSD) and waiting time distribution (WTD) for the random map Eq. (6) where the slopes are sampled from the uniform noise distribution Eq. (8); see the inset on the left for an illustration. All symbols in the plots are generated from computer simulations. (left) MSD $\langle x_t^2 \rangle$ for $b = 2.5, 2.45, 2.38, 2.36384, 2.35, 2.33, 2.3$ (top to bottom) for an ensemble of particles, where each particle experiences a different random sequence. In complete analogy to Fig. 2(a) of the main text there is a characteristic transition between normal diffusion and localization via subdiffusion at a critical $b_c \simeq 2.36384$. (right) The WTD $\eta(t)$ at $b_c \simeq 2.36384$ showing a power law with exponent $-3/2$, cp. with Fig. 2(c) in the main text.

Choosing $a = 0.1$ and keeping it fixed by varying the upper bound b, we obtain the critical parameter b_c at which $\lambda_{unif} = 0$ to $b_c \simeq 2.36384$. As for dichotomic noise we expect the resulting dynamics at b_c to be subdiffusive, which is confirmed in Fig. 1. We conclude that our main result of subdiffusion in a random map is robust against generalising the noise distribution from a dichotomic one to uniform noise on a bounded interval.

III. RANDOM MAP WITH LOG-NORMAL DISTRIBUTION OF SLOPES

Our third model is defined by sampling the slopes in the random map Eq. (6) from a log-normal distribution,

$$\xi \sim \chi_{logn}(\xi) = \text{Lognormal}(\mu,1) = \frac{1}{\xi\sqrt{2\pi}} \exp\left(-\frac{(\ln\xi-\mu)^2}{2}\right) \quad . \tag{10}$$

The Lyapunov exponent can again be calculated from Eq. (7) to

$$\lambda_{logn} = \int_0^\infty d\xi \, \frac{1}{\xi\sqrt{2\pi}} \exp\left(-\frac{(\ln\xi-\mu)^2}{2}\right) \ln\xi = \mu. \tag{11}$$

For $\lambda_{logn} = 0$ this yields trivially a critical parameter value of $\mu_c = 0$ at which we expect the random map to generate subdiffusion. This is indeed confirmed in Fig. 2. We conclude that our main result of subdiffusion in a random map is also robust against generalising the noise distribution from uniform noise on a bounded interval to non-uniform noise on an unbounded interval.

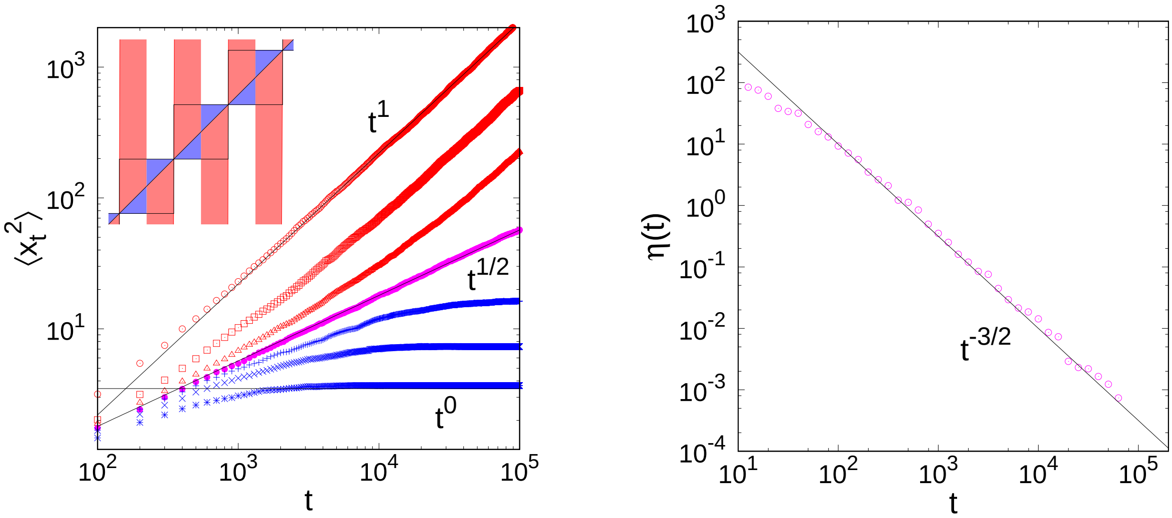

FIG. 2: MSD and WTD for the random map Eq. (6) where the slopes are sampled from the log-normal distribution Eq. (10); see the inset on the left for an illustration. All symbols in the plots are generated from computer simulations. (left) MSD $\langle x_t^2 \rangle$ for $\mu = 0.1, 0.03, 0.01, 0, -0.07, -0.015, -0.03 =$ (top to bottom) for an ensemble of particles, where each particle experiences a different random sequence. In complete analogy to Fig. 2(a) of the main text there is a characteristic transition between normal diffusion and localization via subdiffusion at a critical $\mu = 0$. (right) The WTD $\eta(t)$ at $\mu = 0$ showing a power law with exponent $-3/2$, cp. with Fig. 2(c) in the main text.

IV. RANDOM MAP WITH CHAOTIC TRAPPING

We now investigate whether the vanishing of the Lyapunov exponent exploited in the examples studied above is really necessary for having subdiffusion in our random maps. For this purpose we introduce a fourth model which is similar to our very first model Eqs. (4),(6) but for which we choose two slopes at which the corresponding deterministic maps are both expanding, $a_{loc} = 3/2$ and $a_{exp} = 4$. Note that for $a_{loc} = 3/2$ the deterministic map Eq. (1) in the main text has a positive Lyapunov exponent $\lambda(3/2) = \ln 3/2$, see our calculation at the beginning of Sec. 1. Nevertheless, this map does not generate any long-time diffusion, since particles cannot escape from any unit interval due to the fact that the map does not exceed the unit interval. Hence these sets are trivially decoupled by generating chaotic trapping of particles on any unit interval.

Figure 3 demonstrates that for this purely chaotic model there is no transition between normal diffusion and localisation via subdiffusion. In marked contrast to all our previous models we see that for $p > 0$ the map always generates normal diffusion in the long time limit. Instead of subdiffusion the MSD displays longer and longer transients for shorter times at which diffusion is more and more slowed down when approaching the strictly localised dynamics at $p = 1$. This numerical finding is confirmed by a WTD that is exponential even very close to the localisation value at

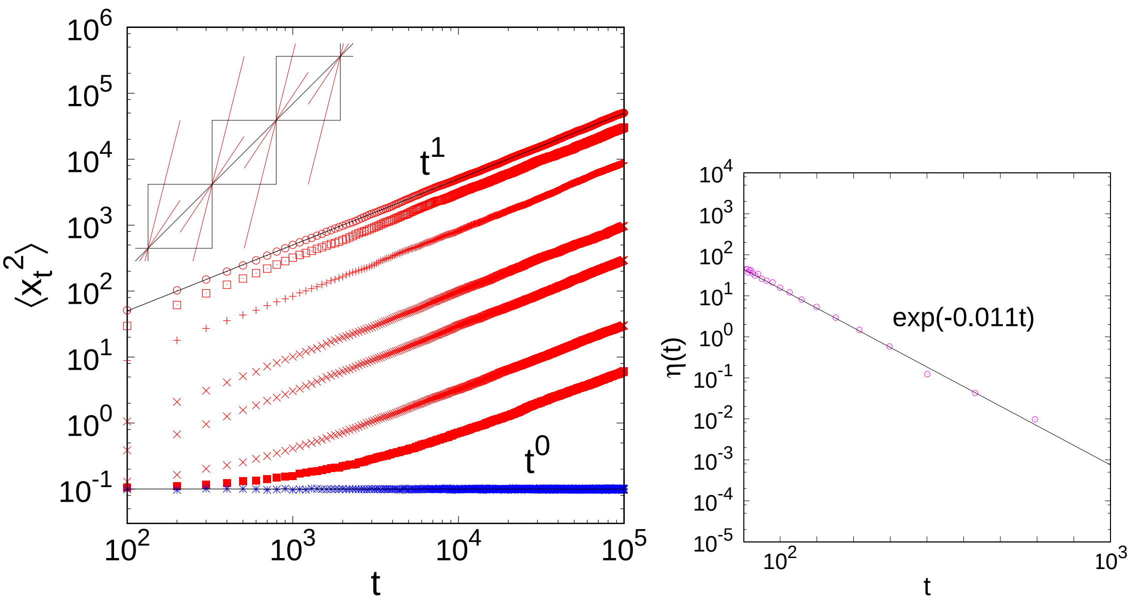

FIG. 3: MSD and WTD for the random map Eq. (6) where the slopes are sampled from the dichotomic distribution Eq. (4) with slopes $a_{loc} = 3/2$ and $a_{exp} = 4$; see the inset on the left for an illustration. Thus, in contrast to the map used for the corresponding results presented in Fig. 2 of the main text here the localisation is not due to contraction onto a stable fixed point but the dynamics remains chaotic on a bounded interval; see the text for further explanations. (left) MSD $\langle x_t^2 \rangle$ for $p = 0.0, 0.5, 0.9, 0.99, 0.995, 0.9995, 0.9999, 1.0$ (top to bottom) for an ensemble of particles, where each particle experiences a different random sequence. In marked contrast to Fig. 2 of the main text, Fig. 1 and Fig. 2 above there is no characteristic transition between normal diffusion and localization via subdiffusion. (right) The WTD $\eta(t)$ at $p = 0.99$ showing an exponential distribution, which is again in sharp contrast to the corresponding power law WTDs in the three figures mentioned above.

$p = 1$ as also shown in Fig. 3. It is in fact well known that for chaotic dynamical systems characterised by a positive Lyapunov exponent the WTD is always exponential while for non-chaotic systems exhibiting regular dynamics it decays as a power law [8, 9]. For an exact analytical calculation of the exponential WTD in an open version of the deterministic map Eq. (1) (main text) we refer to Refs. [1, 4, 6]. To explicitly calculate the WTDs for this and our other random maps, possibly along these lines, in order to quantitatively confirm our numerical results from first principles remains an interesting open problem. We thus conclude that the condition of having a zero Lyapunov exponent is indispensable for observing subdiffusion in these random maps. This is deeply rooted in a mechanism yielding anomalous diffusion that requires an intricate interplay between contracting and expanding dynamics mixed in time.

V. RANDOM NONLINEAR MAP

We finally test whether the observed subdiffusion in random maps also persists in nonlinear maps. As a fifth model we thus replace the map $M_a(x)$ Eq. (1) in the main text by the climbing sine map,

$$C_c(x) = x + c\sin(2\pi x)\ ,\ x \in \mathbb{R} \quad , \tag{12}$$

sketched in the inset of Fig. 4(b) [10]. This map can be derived from discretizing a driven nonlinear pendulum equation in time [11] thus representing a much more general class of nonlinear dynamics than piecewise linear maps. It was found that C_c displays three different regimes of diffusion under parameter variation with an exponent of the MSD of $\alpha = 0$, 1 or 2 [12, 13]. These regimes correspond to different parameter regions in the map's bifurcation diagram reproduced in Fig. 4(a): $\alpha = 1$ occurs in the chaotic regions while $\alpha = 0$ and 2 match to two different classes of periodic windows, where all particles either converge onto attracting localized periodic orbits, or onto ballistic ones.

Figures 4(b) and (c) show numerical results for the MSD and the WTD of the climbing sine map randomized by using the scheme of Fig. 1 in the main text. That is, with probability $1-p$ we choose the parameter $c_1 = 0.8$ sampling the chaotic region while with probability p we draw from the attractive periodic orbit case at $c_2 = 0.2$. Both figures display results for the critical probability $p_c \simeq 0.505702$, where the map's Lyapunov exponent is approximately zero. One can clearly see excellent matching of the exponents $\sim t^{1/2}$ for the MSD and $\sim t^{-3/2}$ for the WTD as predicted

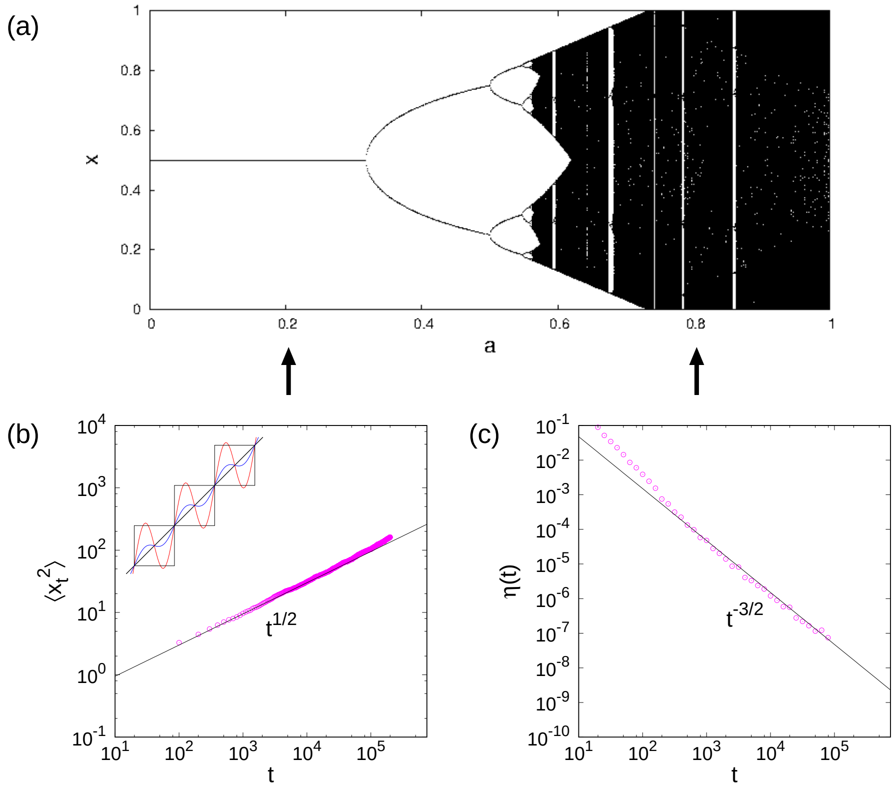

FIG. 4: Subdiffusion in the random climbing sine map. (a) The bifurcation diagram of the climbing sine map depicted in the inset of (b) [12, 13]. The two arrows denote the two specific parameter values $c_1 = 0.8$ and $c_2 = 0.2$ chosen with probability $1 - p$, respectively p, to generate a random dynamical system from this map as explained in the text. (b) MSD at the critical probability $p_c \simeq 0.505702$ where the random map's Lyapunov exponent is approximately zero. The symbols are from simulations, the line is a fit based on CTRW theory. The MSD clearly exhibits subdiffusion with exponent 1/2. (c) The WTD for the same p_c showing a power law with exponent $-3/2$ as predicted by CTRW theory; symbols and lines are as in (b).

by conventional CTRW theory, cf. Fig. 2 in the main text. Hence, the basic mechanism that we propose to generate anomalous diffusion due to randomization of deterministic dynamics is also robust for a generic nonlinear map.

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