MTH4100 Calculus I

Lecture notes for Week 10

Thomas' Calculus, Sections 5.2 to 5.5

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example:

$$\begin{aligned}\sum_{k=1}^{n} k &= 1 && +2 && +3 && +\dots && +(n-1) && +n \\ &= n && +(n-1) && +(n-2) && +\dots && +2 && +1\end{aligned}$$

$\Rightarrow 2\sum_{k=1}^{n} k = n(n+1)$ (C.F.Gauß, $\simeq$ 1784), or

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

The first n squares:	$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
The first n cubes:	$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

Can be proved by mathematical *induction*, see textbook Appendix 1.

Limits of finite sums

example: Compute the area R below the graph of $y = 1 - x^2$ and above the interval $[0, 1]$.

- Subdivide the interval into n subintervals of width $\Delta x = \frac{1}{n}$:

$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \left[\frac{2}{n}, \frac{3}{n}\right], \dots, \left[\frac{n-1}{n}, \frac{n}{n}\right].$$

- Choose the lower sum: $c_k = \frac{k}{n}$, $k \in \mathbb{N}$ is the rightmost point.
- Do the summation:

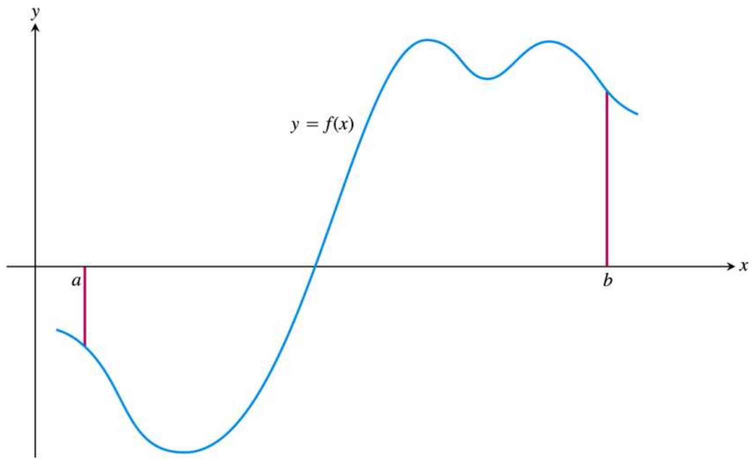
$$f\left(\frac{1}{n}\right)\frac{1}{n} + f\left(\frac{2}{n}\right)\frac{1}{n} + \dots + f\left(\frac{n}{n}\right)\frac{1}{n} = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n} = \sum_{k=1}^{n}\left(1 - \left(\frac{k}{n}\right)^2\right)\frac{1}{n} =$$
$$= \sum_{k=1}^{n}\left(\frac{1}{n} - \frac{k^2}{n^3}\right) = \frac{1}{n}\sum_{k=1}^{n} 1 - \frac{1}{n^3}\sum_{k=1}^{n} k^2 = \frac{1}{n}n - \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} = 1 - \frac{2n^3 + 3n^2 + n}{6n^3} =$$
$$= \frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}$$

- Lower sum: $R \geq \frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}$.
- Upper sum: $R \leq \frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2}$. (exercise)
- As $n \to \infty$, both sums converge to $\frac{2}{3}$. Therefore, $R = \dfrac{2}{3}$.

note: Any other choice of c_k would give the same result. (why?)

Riemann sums and definite integral

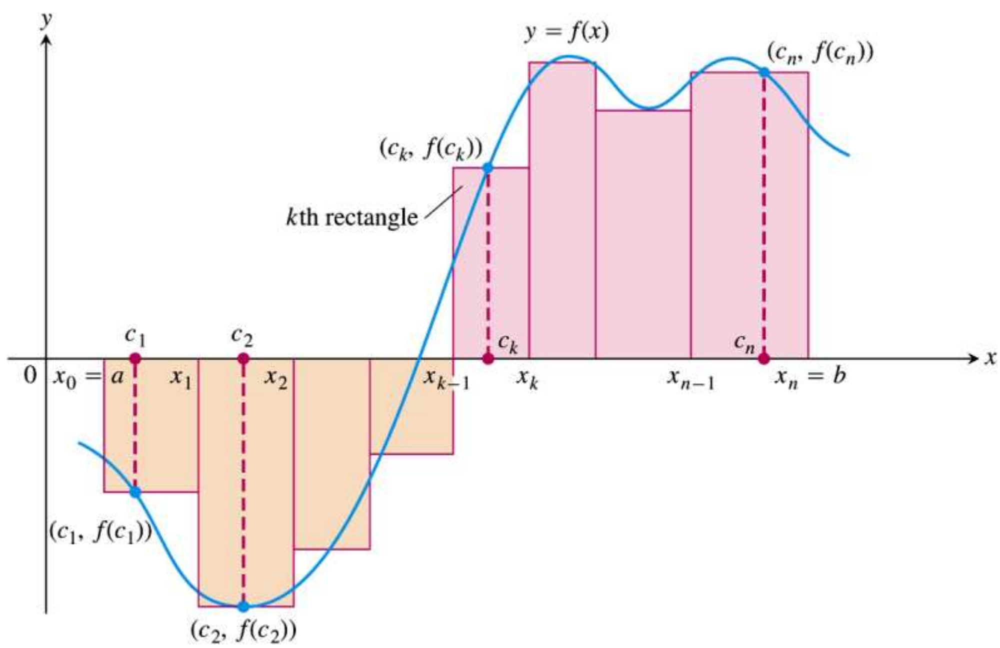
Consider a typical continuous function over $[a, b]$:



Partition $[a, b]$ by choosing $n - 1$ points between a and b:

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b\,.$$

Note that $\Delta x_k = x_k - x_{k-1}$, the width of the subinterval $[x_{k-1}, x_k]$, may vary.
Choose $c_k \in [x_{k-1}, x_k]$ and construct rectangles:



The resulting sums $S_p = \sum_{k=1}^{n} f(c_k)\Delta x_k$ are called *Riemann sums for f on $[a, b]$*.
Then choose finer and finer partitions by taking the limit such that the width of the *largest subinterval* goes to zero.
For a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ we write $||P||$ (called "norm") for the width of the largest subinterval.

DEFINITION **The Definite Integral as a Limit of Riemann Sums**

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number I is the **definite integral of f over $[a, b]$** and that I is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

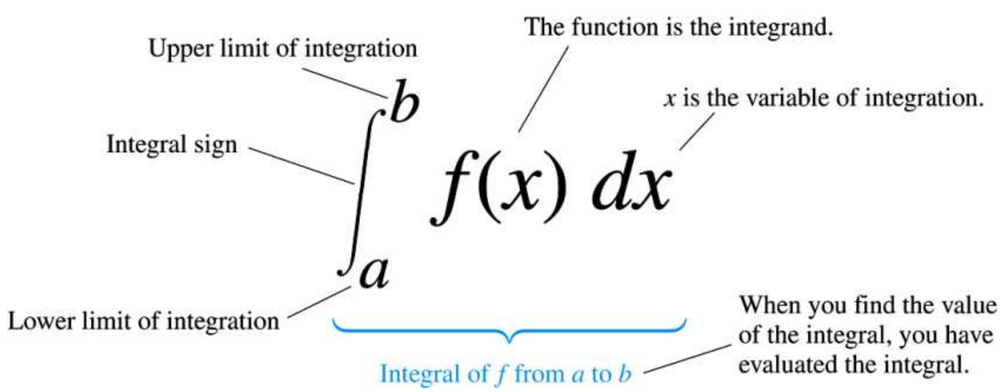
Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - I \right| < \epsilon .$$

shorthand notation:

$$I = \lim_{||P|| \to 0} \sum_{k=1}^{n} f(c_k) \Delta x_k = \int_a^b f(x) dx$$

with



note:

$$\int_a^b f(t)dt = \int_a^b f(x)dx \,, \text{ etc.}$$

THEOREM 1 **The Existence of Definite Integrals**

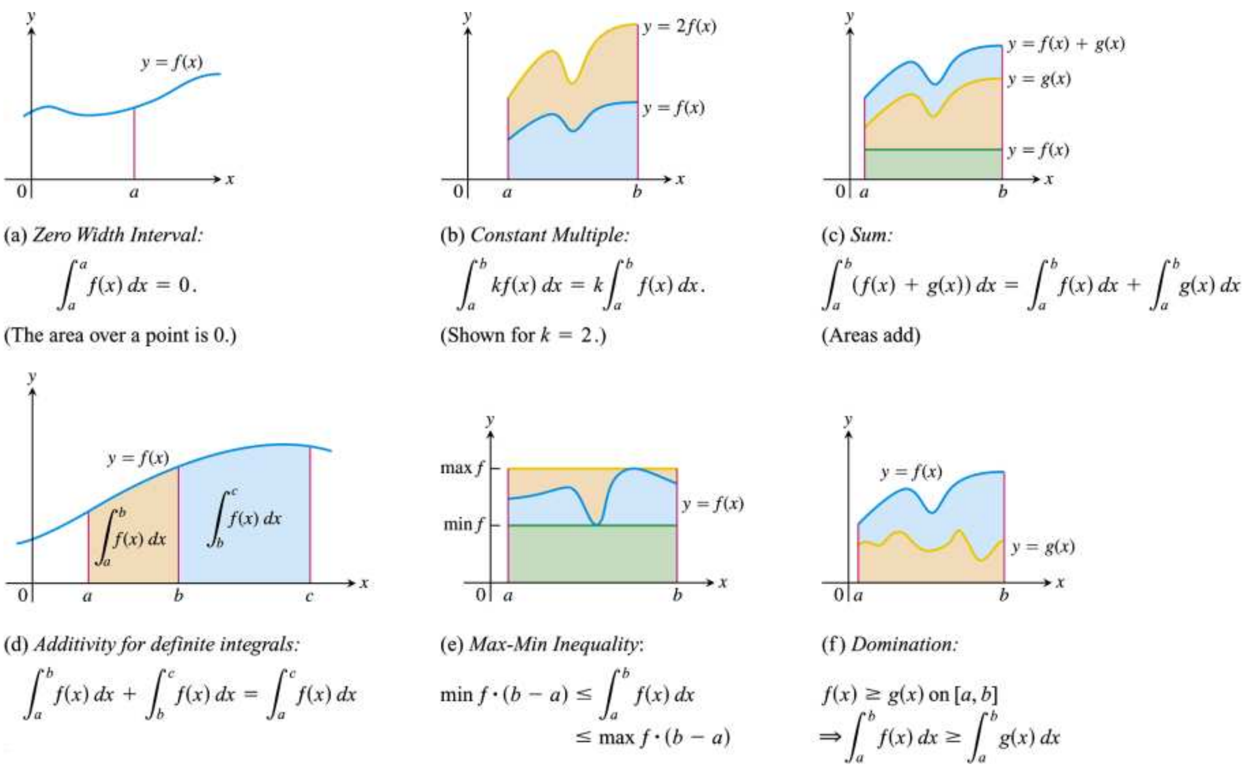
A continuous function is integrable. That is, if a function f is continuous on an interval $[a, b]$, then its definite integral over $[a, b]$ exists.

(idea of proof: check convergence of upper/lower sums; see p.345 of book for further details)

example of a nonintegrable function on $[0, 1]$: $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

Upper sum is always 1; lower sum is always 0 $\Rightarrow \int_0^1 f(x)dx$ does not exist!

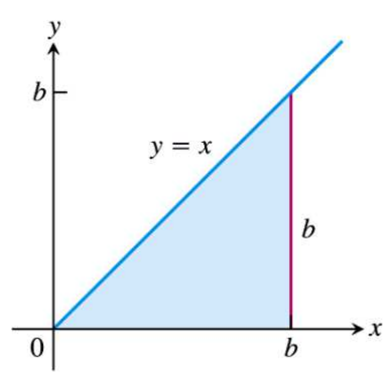
Theorem 2 *For integrable functions f, g on $[a, b]$ the definite integral satisfies the following rules:*



and *(g) order of integration:* $\displaystyle\int_b^a f(x)dx = -\int_a^b f(x)dx$ *(for idea of proof of (b) to (f) see book p.348; (a), (g) are definitions!)*

Area under the graph and mean value theorem

example: $f(x) = x$, $a = 0$, $b > 0$



Area $A = \frac{1}{2}b^2$. Definition of integral: Choose $x_k = kb/n$ with right endpoints c_k.

$$I = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{kb}{n}\cdot\frac{b}{n} = \lim_{n\to\infty}\frac{b^2}{n^2}\sum_{k=1}^{n} k = \lim_{n\to\infty}\frac{b^2}{n^2}\frac{n(n+1)}{2} = \frac{b^2}{2}$$

DEFINITION Area Under a Curve as a Definite Integral

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$

Consider the (arithmetic) *average* of n function values on $[a, b]$:

$$\frac{1}{n}\sum_{k=1}^{n} f(c_k) = \frac{1}{n\Delta x}\sum_{k=1}^{n} f(c_k)\Delta x \to \frac{1}{b-a}\int_a^b f(x)dx\ (n \to \infty)$$

DEFINITION The Average or Mean Value of a Function

If f is integrable on $[a, b]$, then its **average value on** $[a, b]$, also called its **mean value**, is

$$\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

example: $f(x) = x\ ,\ x \in [0, b]$ (see above)

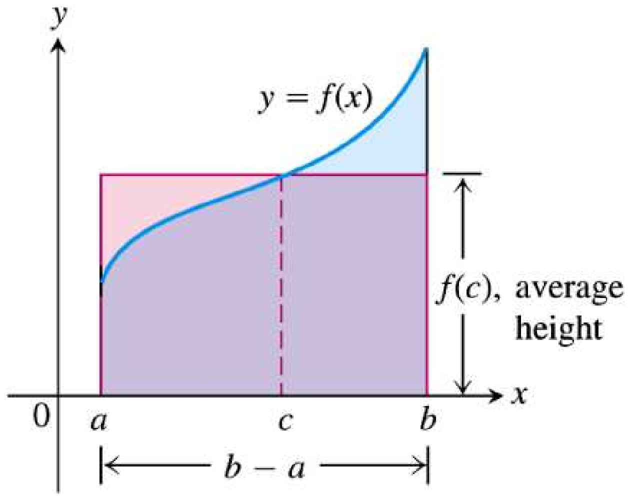
$$\text{av}(f) = \frac{1}{b-0}\int_0^b x dx = \frac{1}{b}\left.\frac{x^2}{2}\right|_0^b = \frac{b^2}{2b} = \frac{b}{2}$$

Theorem 3 (The mean value theorem for definite integrals) *If f is continuous on $[a, b]$, then there is a $c \in [a, b]$ with*

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx\ .$$

Interpretation, loosely speaking: "f assumes its average value somewhere on $[a, b]$."

geometrical meaning:



(proof: see book p.357; not hard; based on max-min-inequality for integrals and intermediate value theorem for continuous functions)

example: Let f be continuous on $[a, b]$ with $a \neq b$ and

$$\int_a^b f(x)dx = 0 .$$

Show that $f(x) = 0$ at least once in $[a, b]$.
Solution: According to the last theorem, there is a $c \in [a, b]$ with

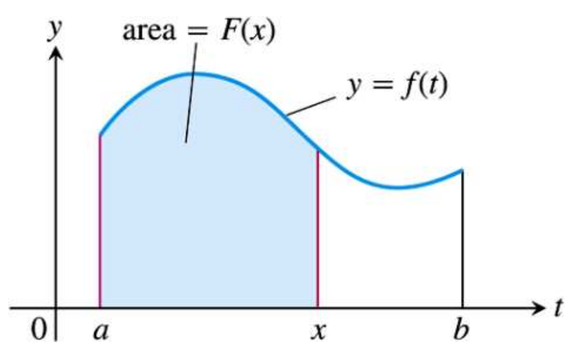
$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx = 0 .$$

The Fundamental Theorem of Calculus

For a continuous function f, define

$$F(x) = \int_a^x f(t)dt .$$

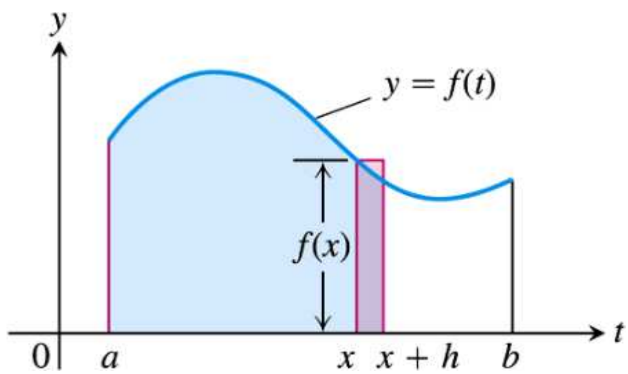
Geometric interpretation:



Compute the difference quotient:

$$\begin{aligned}
\frac{F(x+h)-F(x)}{h} &= \frac{1}{h}\left(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right) \\
\text{(additivity rule and see figure below)} &= \frac{1}{h}\int_x^{x+h} f(t)dt \\
\text{(mean value theorem for definite integrals)} &= f(c)
\end{aligned}$$

for some c with $x \le c \le x+h$.



Since f is continuous,

$$\lim_{h\to 0} f(c) = f(x)$$

and therefore

$$F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = f(x)\ .$$

We have just proven (except a little detail - which one?)

THEOREM 4 The Fundamental Theorem of Calculus Part 1

If f is continuous on $[a, b]$ then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$;

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

examples:

1. $$\frac{d}{dx}\int_a^x \frac{1}{1+4t^3}dt = \frac{1}{1+4x^3}$$

2. Find $$\frac{d}{dx}\int_2^{x^2} \cos t\, dt\ :$$

Define

$$y = \int_2^u \cos t \, dt \text{ with } u = x^2$$

Apply the chain rule:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= \left(\frac{d}{du} \int_2^u \cos t \, dt \right) \cdot \frac{du}{dx} \\
&= \cos u \cdot 2x \\
&= 2x \cos x^2
\end{aligned}$$

Let f be continuous on $[a, b]$. We know that

$$\int_a^x f(t)dt = G(x)$$

is an antiderivative of f, as $G'(x) = f(x)$, see theorem above.
The most general antiderivative is $F(x) = G(x) + C$ on $x \in (a, b)$ (why?). We thus have

$$\begin{aligned}
F(b) - F(a) &= (G(b) + C) - (G(a) + C) \\
&= G(b) - G(a) \\
&= \int_a^b f(t)dt - \int_a^a f(t)dt \\
\text{(zero width interval rule)} &= \int_a^b f(t)dt \, .
\end{aligned}$$

We have just shown (to be amended by considering F, G at the boundary points a, b)

THEOREM 4 (Continued) **The Fundamental Theorem of Calculus Part 2**

If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

Recipe to calculate $\int_a^b f(x)dx$:

1. Find an antiderivative F of f
2. Calculate $F(b) - F(a)$

Notation:

$$F(b) - F(a) = F(x)|_a^b$$

example:

$$\begin{aligned}\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx &= \left(x^{3/2} + \frac{4}{x}\right)\Bigg|_1^4 \\ &= \left(4^{3/2} + \frac{4}{4}\right) - \left(1^{3/2} + \frac{4}{1}\right) \\ &= 4\end{aligned}$$

Fundamental Theorem of Calculus: summary

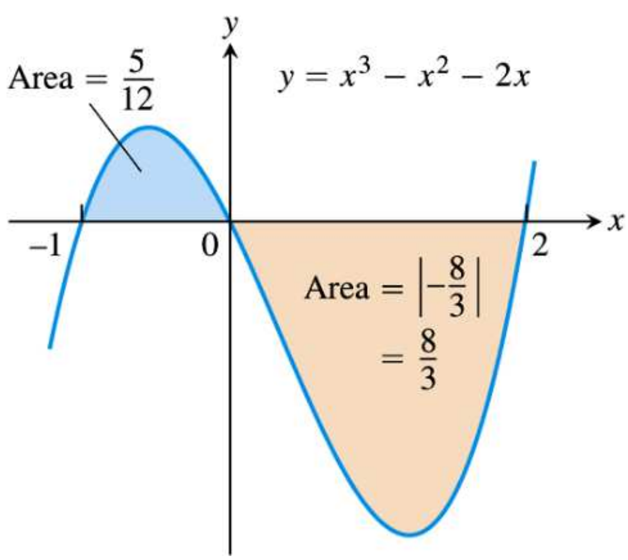
$$\frac{d}{dx}\int_a^x f(t)dt = \frac{dF}{dx} = f(x)$$

$$\int_a^x f(t)dt = \int_a^x \frac{dF}{dt}dt = F(x) - F(a)$$

Processes of integration and differentiation are "inverses" of each other!

Finding total areas

example:



To find the *area* between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$, do the following:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate over each subinterval.
3. Add the *absolute* values of the integrals.

example continued:

$$f(x) = x^3 - x^2 - 2x \, , \; -1 \leq x \leq 2$$

1. $f(x) = x(x^2 - x - 2) = x(x+1)(x-2)$: zeros are -1, 0, 2
2. $$\int_{-1}^{0} (x^3 - x^2 - 2x)dx = \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2\right)\Big|_{-1}^{0} = \frac{5}{12}$$ $$\int_{0}^{2} (x^3 - x^2 - 2x)dx = \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2\right)\Big|_{0}^{2} = -\frac{8}{3}$$
3. $A = \left|\frac{5}{12}\right| + \left|-\frac{8}{3}\right| = \frac{37}{12}$

The substitution rule

motivation: develop more general techniques for calculating antiderivatives
Recall the chain rule for $F(g(x))$:

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x)$$

If F is an antiderivative of f, then

$$\frac{d}{dx}F(g(x)) = f(g(x))g'(x)$$

Now compute

$$\begin{aligned} \int f(g(x))g'(x)dx &= \int \left(\frac{d}{dx}F(g(x))\right) dx \\ \text{(fundamental theorem)} &= F(g(x)) + C \\ (u = g(x)) &= F(u) + C \\ \text{(fundamental theorem)} &= \int F'(u)du \\ &= \int f(u)du \end{aligned}$$

We have just proved

THEOREM 5 The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

method for evaluating

$$\int f(g(x))g'(x)dx \ :$$

1. Substitute $u = g(x)$, $du = g'(x)dx$ to obtain $\int f(u)du$.
2. Integrate with respect to u.
3. Replace $u = g(x)$.