Queen Mary
University of London

MAS 424 **Introduction to Dynamical Systems**

M.Sci. Examination

Thursday 19th May 2005, 10.00 am **Duration: 3 hours**

You should attempt all questions. Marks awarded are shown next to the question. Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offense.

1. *Stability analysis*
   Consider one-dimensional maps $F : \mathbb{R} \to \mathbb{R}$, $x_{n+1} = F(x_n)$, $n \in \mathbb{N}$.

   (a) (3 marks) Define the following concepts: a fixed point, a periodic point, a periodic orbit.

   (b) (12 marks) Consider the map $C(x) = 2x^2 - 5x$, $x \in \mathbb{R}$. Calculate the set $Per_2(C)$ of all period 2 points for this map. Draw the graph of $C(x)$ and mark the positions of all period 2 points. Include cobweb plots for all period two orbits and illustrate the stability of the fixed points by cobweb plots.

   (c) (3 marks) State (without proof) a criterion from which one can calculate the stability of a periodic point. What is a prime period?

   (d) (4 marks) Use your stability criterion in order to classify the stability of all fixed points and of all prime period two orbits of the map $C(x)$.

   (e) (4 marks) For maps of the type of F given above, define what is meant by the (local) Lyapunov exponent $\lambda(x)$ at some point x. Calculate $\lambda(x)$ for all fixed points and for all prime period two orbits of $C(x)$.

   (f) (10 marks) Let the map F be continuous. State the Intermediate Value Theorem and use it to prove that if F has a periodic point of prime period two F also has a fixed point. (*hint:* the prime period two orbit defines an interval on which you may work in analogy to the proof of the Fixed Point Theorem.)

2. *Chaos and periodic orbits*

   (a) (6 marks) Show that for the map $B_3(x) = 3x \bmod 1$, $x \in [0, 1)$, the number of periodic orbits of period n is $|Per_n(B_3)| = 3^n - 1$, as follows. Draw $B_3(x)$ and its second iterate. Identify $Fix(B_3)$ and $Per_2(B_3)$ in your drawings and calculate the corresponding periodic points analytically. On this basis, argue for the result for general n. A proof by induction is not required.

(b) (9 marks) Consider the map

$$D(x) = \begin{cases} 2x+1 & , \quad -1 \le x < -1/2 \\ 2x & , \quad -1/2 \le x < 1/2 \\ 2x-1 & , \quad 1/2 \le x \le 1\,. \end{cases}$$

Draw the graph of this map. Is $D(x)$ expanding or piecewise expanding? Is it hyperbolic or piecewise hyperbolic? Calculate its Lyapunov exponent $\lambda\ \forall x$. Is $D(x)$ sensitive? Has it a dense set of periodic orbits (here no calculation required, you may refer to known results)? Is it topologically transitive? Justify your answers.

(c) (3 marks) What is chaos in the sense of Devaney, Robinson and Lyapunov? Is $D(x)$ chaotic according to any of these definitions? Is the restricted map $D(x) : [-1, 0] \to [-1, 0]$ chaotic?

(d) (4 marks) State the Poincaré-Bendixson theorem both for time-continuous and for time-discrete dynamical systems. Apply it to the following two examples:

i. (3 marks) the differential equation (Duffing oscillator) $\ddot{x} + \nu\dot{x} + x^3 - x = A\sin(\omega t)\ ,\ x \in \mathbb{R}\,,\ t \ge 0$, where $\nu > 0, A > 0, \omega > 0$ are all constant. According to Poincaré-Bendixson, is chaos possible in this system?

ii. (5 marks) the map $x_{n+1} = S(x_n) = x_n + k\sin(x_n)\,,\ n \in \mathbb{N}\,,\ x_n \in \mathbb{R}$, where $k \ge 0$ is a parameter. Determine the range of k for which chaos is possible, according to Poincaré-Bendixson. Justify your answer by drawing graphs of $S(x)$ at representative values of k.

3. *Topological conjugacy*
Let $M : I \to I\ ,\ N : J \to J\ ,\ I, J \subset \mathbb{R}$, be one-dimensional maps.

(a) (2 marks) Define what is meant by a topological conjugacy H from N to M. Summarise your definition in a diagram depicting the relations between these three maps.

(b) (4 marks) Let

$$U : [-1, 1] \to [-1, 1]\,,\ U(x) = 1 - 2x^2$$

and

$$T : [0, 1] \to [0, 1]\,,\ T(y) = \begin{cases} 2y & , \quad 0 \le y \le -1/2 \\ 2-2y & , \quad 1/2 < y \le 1\,. \end{cases}$$

Prove that $x = \tilde{H}(y) = -\cos(\pi y)\ ,\ y \in [0, 1]$, is a topological conjugacy for both maps.

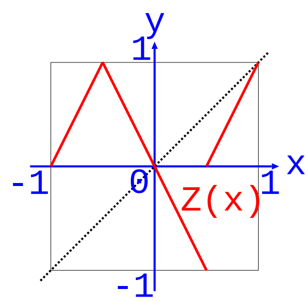
(c) (4 marks) Prove that x is a periodic point of period 2 for M if and only if $H(x)$ is a periodic point of period two for N.

(d) (8 marks) Consider the Bernoulli shift $B_2(x) = 2x \bmod 1$, $x \in [0, 1)$. Define the binary representation for $x \in [0, 1)$. How do periodic points of $B_2(x)$ show up in this representation? Give an example. By using this binary representation, show that $|Per_n(B_2)| = 2^n - 1$ (no proof by induction required).

4. *Frobenius-Perron equation and topological transition matrices*

(a) (2 marks) Write down the Frobenius-Perron equation for a time-discrete dynamical system and explain its meaning.

(b) (4 marks) Consider the map $Z(x)$ represented by the sloping bold lines below:



The quadratic grid included in this figure defines a Markov partition, the dotted diagonal is a guide to the eye. State verbally what it means to say that a partition is Markov. Define a suitable alphabet on the Markov partition above and represent the action of $Z(x)$ on this alphabet in a Markov graph.

(c) (2 marks) With respect to this partition, write down the topological transition matrix $\underline{\underline{T}}$ for $Z(x)$. State the equivalent of the Frobenius-Perron equation for $\underline{\underline{T}}$.

(d) (6 marks) By solving the eigenvalue problem of $\underline{\underline{T}}$, calculate the invariant density $\varrho^*(x)$ for $Z(x)$, which is a piecewise constant function that you may write as a column vector $\underline{\varrho}^*$ defined over the Markov partition. Note that you should normalise $\underline{\varrho}^*$.

(e) (2 marks) Define what is meant by the Lyapunov exponent λ of a one-dimensional map acting on the real line with respect to the invariant probability density of the map. Calculate λ for $Z(x)$.