Queen Mary
University of London

MAS424/MTHM021 **Introduction to Dynamical Systems**

Exercise Sheet 3 **Rainer Klages**

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1. *Ljapunov exponents*

   (a) Consider again the following maps:

      i. $C(x) := -x^2 + x + 2$ from problem 3.(a) of exercise sheet 1

      ii. $D(x) := 2x^2 - 5x$ from problem 1.(b) of exercise sheet 2

   Calculate the Ljapunov exponents for all fixed points and for all prime period two orbits of these maps.

   (b) Suppose the map $F(x)$ acting on the real line is of class C^1. Prove that if F is expanding or contracting it is hyperbolic. Prove that if F has a positive Ljapunov exponent $\lambda(x)$ for all x it is sensitive (*hint:* generalise the proof for the Bernoulli shift shown in the lecture). Prove that F expanding implies sensitivity of F.

2. *Definitions of chaos*

   (a) Consider the map $R(x)$ defined in problem 3.(b) on exercise sheet 2. Calculate the Ljapunov exponent of this map for all x, as far as this is possible. Is $R(x)$ chaotic according to the definitions of Devaney, Wiggins and Ljapunov? Justify your answers.

   (b) Consider the map $E(x)$ defined in problem 3.(a) on exercise sheet 2.

      i. Calculate the Ljapunov exponent of this map for all x, as far as this is possible. Is $E(x)$ sensitive?

      ii. Is $E(x)$ chaotic according to the definitions of Devaney, Wiggins and Ljapunov? Is the restricted map $E(x) : [-1, 0] \to [-1, 0]$ chaotic? Justify your answers. Previous results of coursework or lectures may be used for this purpose.

3. *Frobenius-Perron equation*
   Consider the Frobenius-Perron equation

$$\rho_{n+1}(x) = \sum_{x=M(x^i)} \rho_n(x^i)|M'(x^i)|^{-1} =: P\rho_n(x)$$

   for a map $M(x)$ defined on the real line, where $\rho_n(x)$ are probability densities at time step $n \in \mathbb{N}_0$ and P defines the Frobenius-Perron operator.

   (a) Verify the statement given in the lecture that P is a linear and positive operator.

(b) Construct P for the map $B_3(x) = 3x \bmod 1$ and verify that $\rho^*(x) = 1$ is an invariant density of the corresponding Frobenius-Perron equation.

(c) Consider the map defined by $U(x) = 1 - 2x^2$, $x \in [-1, 1]$ (Ulam map). Draw the graph of $U(x)$, construct its Frobenius-Perron operator and verify that $\varrho^*(x) = 1/(\pi\sqrt{1-x^2})$ is an invariant density of the map.

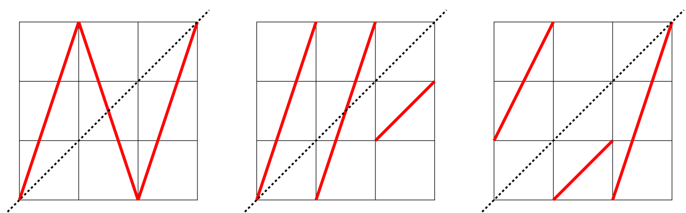
(d) Construct P for the map $F(x)$ defined in the lecture and verify that

$$\rho^*(x) = \begin{cases} 4/3 & , \quad 0 \leq x < 1/2 \\ 2/3 & , \quad 1/2 \leq x \leq 1 \end{cases}$$

is an invariant density of the corresponding Frobenius-Perron equation.

4. *Markov graphs*

(a) Encode the three maps shown below by suitable sets of symbols and construct their Markov graphs.



The maps are shown as bold (red) lines, the grids define Markov partitions for these maps, the dotted diagonals are guides to the eye.

(b) If the dynamics defined by a Markov graph generates all possible combinations of the associated set of symbols, one speaks of a *full shift*. If the resulting symbol sequences are restricted to a subspace, i.e., certain symbol sequences are forbidden, one speaks of a *subshift of finite type*. Decide whether the above maps represent full shifts or subshifts of finite type.

Model solutions will be on the course webpage starting from Wednesday, November 29th, 2007.