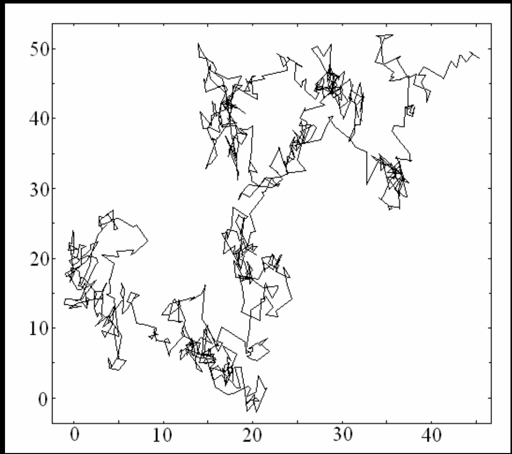

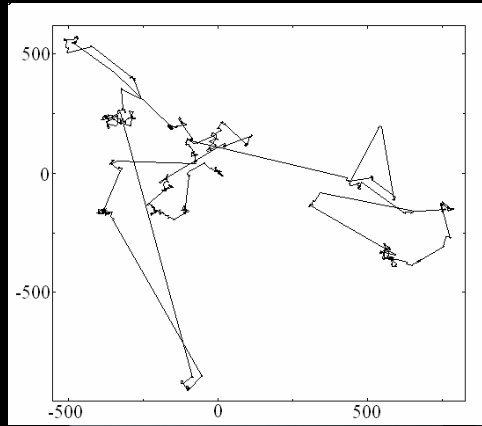

QMUL, 10 Jul 2009

A Short Introduction to the Theory of Lévy Flights

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and

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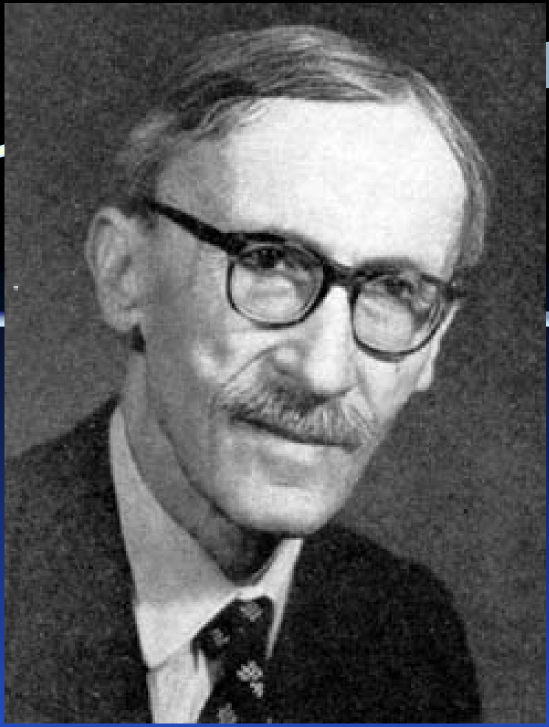
LÉVY motion(-s) $\rightarrow$ "LÉVY flight(-s)"
P. Lévy $\rightarrow$ B. Mandelbrot:
Paradigm of non-Brownian random motion

Mathematical Foundations:

Brownian Motion	Lévy Motion
1. Central Limit Theorem	1. Generalized Central Limit Theorem
2. Properties of Gaussian probability laws	2. Properties of α-stable Lévy probability laws



Paul Pierre Lévy (1886-1971)

Theory of Lévy stable probability distributions in mathematical monographs:

Lévy (1925, 1937)
Khintchine (1938)
Gnedenko & Kolmogorov (1949-53)
Feller (1966-71)
Lukacs (1960-70)
Zolotarev (1983-86, 1997)
Janiki & Weron (1994)
Samorodnitski & Taqqu (1994)

B.V. Gnedenko, A.N. Kolmogorov, "Limit Distributions for Sums of Independent Random Variables" (1949)

The prophecy: "All these distribution laws, called stable, deserve the most serious attention. It is probable that the scope of applied problems in which they play an essential role will become in due course rather wide."(but did not mention any !)

The guidance for those who write books on statistical physics: "… "normal" convergence to abnormal (that is, different from Gaussian –A. Ch.) stable laws … , undoubtedly, has to be considered in every large textbook, which intends to equip well enough the scientist in the field of statistical physics."(fragmentary exposition in Balescu, Ebeling&Sokolov,...)

First remarkable property of stable distributions

1. **Stability:** distribution of sum of independent identically distributed stable random variables = distribution of each variable (up to scaling factor)

$$X_1 + X_2 + \;.\;.\;.\;. + X_n = \sum_{i=1}^{n} X_i \overset{d}{=} c_n X$$

$c_n = n^{1/a}$, $0 < \alpha \leq 2$, α **is Lèvy index**

Gauss: $\alpha = 2$, $c_n = n^{1/2}$

Corollary 1. Statistical self-similarity, random fractal:

When does the whole look like its parts ⇒ description of random fractal ptocesses

Corollary 2. Normal diffusion law, Brownian motion

$$\Delta X \sim \sum_{i=1}^{n} X_i \propto n^{1/2} \propto t^{1/2}$$

or the rule of summing variances

$$\left\langle (\Delta X)^2 \right\rangle = \sum_{i=1}^{n} \left\langle X_i^2 \right\rangle = \left\langle X^2 \right\rangle n = \left\langle X^2 \right\rangle t$$

Corollary 3. Superdiffusion, Lévy

$$\Delta X \sim \sum_{i=1}^{n} X_i \propto n^{1/\alpha} \propto t^{1/\alpha}, \quad 1/\alpha > 1/2$$

SECOND remarkable property of stable distributions

2. Power law asymptotics ~ $|x|^{(-1-\alpha)}$ ("heavy tails")

$$\hat{p}_X(k;\alpha,D) \equiv \langle \exp(ikX) \rangle = \exp\left(-D|k|^{\alpha}\right),\ 0<\alpha\le 2, D>0 \underset{0<\alpha<2}{\Rightarrow} p_X(x,\alpha,D) \underset{x\to\infty}{\propto} \frac{1}{|x|^{1+\alpha}}$$

$$\Rightarrow \quad \langle x^2 \rangle = \infty \quad , \quad 0<\alpha<2 \qquad \langle |x|^q \rangle < \infty, \quad 0<q<\alpha$$

Particular cases

1. Gauss, $\alpha = 2$

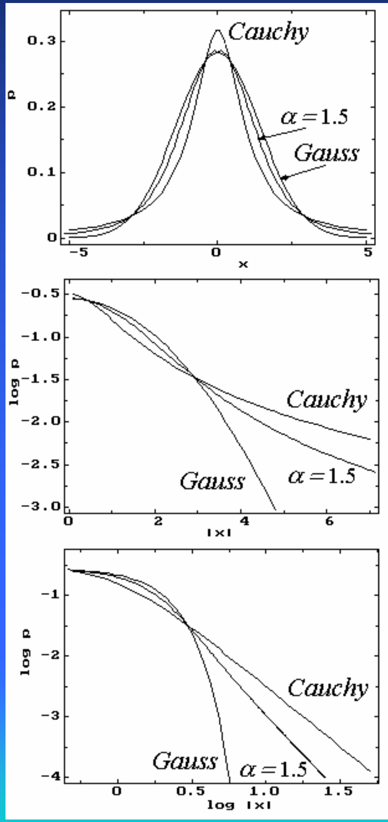
All moments are finite

$$p(x,2,D) = \frac{1}{\sqrt{4\pi D}} \exp\left(-\frac{x^2}{4D}\right)$$

2. Cauchy, $\alpha = 1$

Moments of order q < 1 are finite

$$p(x,1,D) = \frac{D}{\pi\left(D^2 + x^2\right)}$$



Corollary: description of highly non-equilibrium processes possessing large bursts and/or outliers

THIRD remarkable property of stable distributions

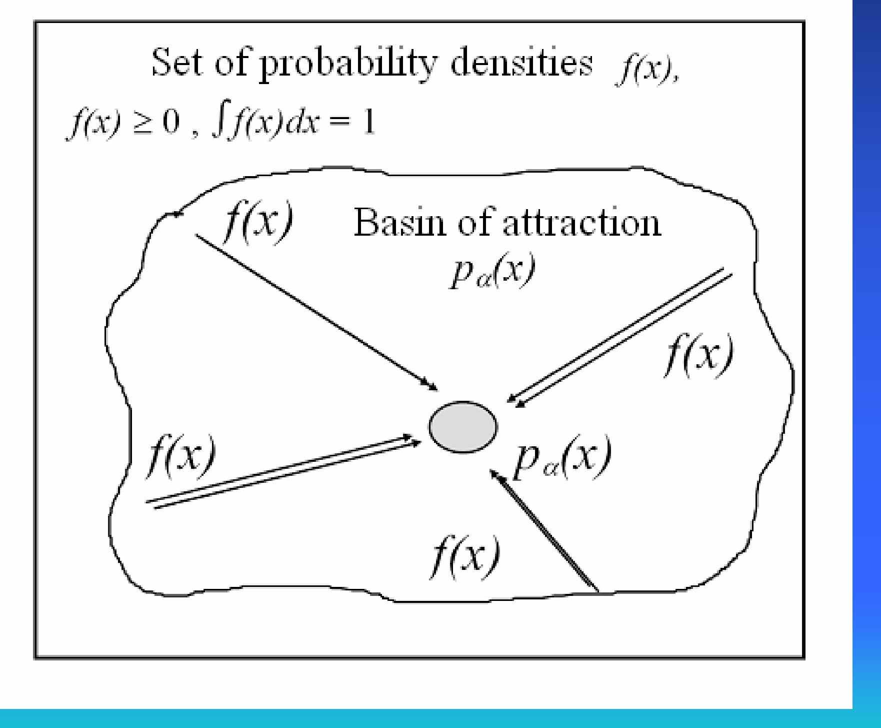
3. <u>Generalized Central Limit Theorem</u>: stable distributions are the limit ones for distributions of sums of random variables (have domain of attraction) ⇒ appear, when evolution of the system or the result of experiment is determined by the sum of a large number of random quantities

Gauss: attracts *f(x)* with <u>finite</u> variance

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 f(x) dx < \infty$$

Lévy: attract *f(x)* with <u>infinite</u> variance and the <u>same</u> asymptotic behavior

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 f(x) dx = \infty \quad , \quad f(x) \sim x^{-1-\alpha}, \quad 0<\alpha<2$$

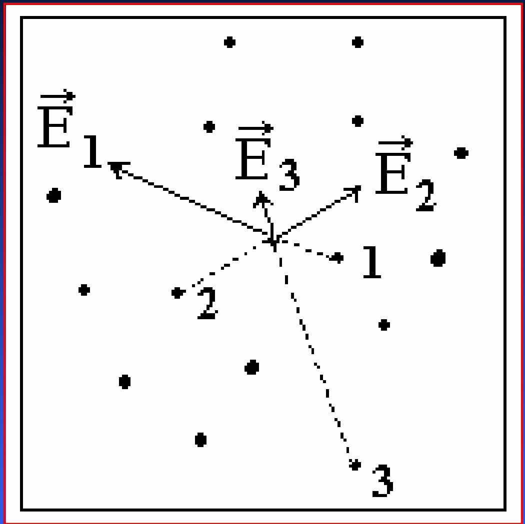


Σ 1

Due to the three remarkable properties of stable distributions it is now believed that the Lévy statistics provides a framework for the description of many natural phenomena in physical, chemical, biological, economical, … systems from a general common point of view

Electric Field Distribution in an Ionized Gas (Holtzmark 1919)

Application: **spectral lines broadening in plasmas**



$$\vec{E}_i = \frac{e\vec{r}_i}{r_i^3}$$

$$n = \frac{N}{V} = const$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \ldots + \vec{E}_n$$

$$W\left(\vec{E}\right) = \int_{-\infty}^{\infty} \frac{d\vec{k}}{(2\pi)^3} \widehat{W}\left(\vec{k}\right) e^{-i\vec{k}\vec{E}}$$

$$\widehat{W}\left(\left|\vec{E}\right|\right) \sim \frac{1}{\left|\vec{E}\right|^{5/2}}$$

3 – dimensional symmetric stable distribution $\alpha = 3/2$

$$\widehat{W}\left(\vec{k}\right) = e^{-a\left|\vec{k}\right|^{3/2}}$$

Related examples:
Systems with Long-Range interactions

- gravity field of stars (=3/2)
- electric fields of dipoles (= 1)
- velocity field of point vortices in fully developed turbulence (=3/2) ……

(to name only a few)

- <u>also</u>: asymmetric Levy stable distributions in the first passage theory, single molecule line shape cumulants in glasses …

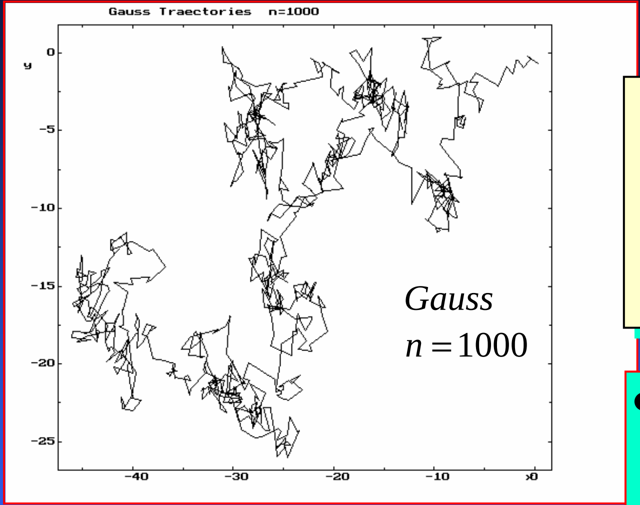
Normal vs Anomalous Diffusion

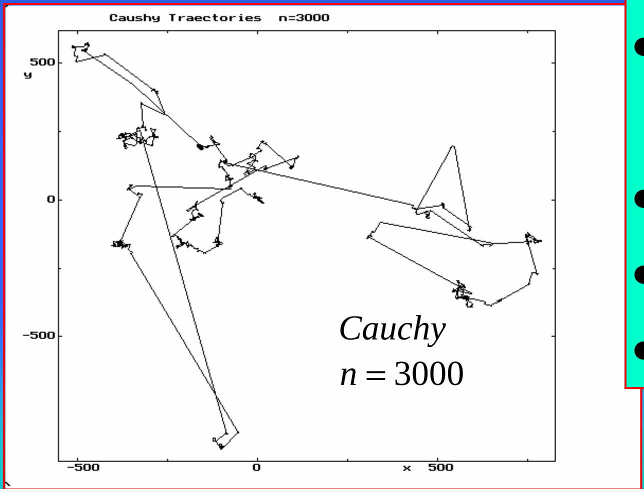
Normal diffusion $\left\langle \left(\vec{R}-\vec{R}_0\right)^2 \right\rangle = c_{\dim} D t \propto t^1$

Bachelier (1900), *Einstein* (1905)

Anomalous diffusion

$$\left\langle \left(\vec{R}-\vec{R}_0\right)^2 \right\rangle \propto t^{\mu},\ \mu \neq 1$$





Superdiffusion

$$\mu > 1$$

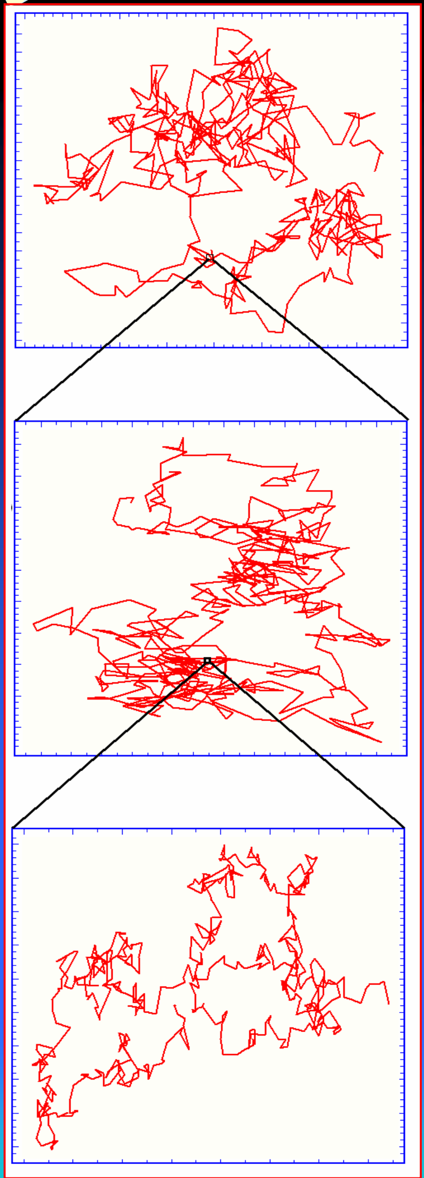
- *turbulent media (gases, fluids, plasmas)*
- *Hamiltonian chaotic systems*
- *deterministic maps*
- *foraging movement*
- *financial markets*

Subdiffusion

$$\mu < 1$$

- *transport on fractals*
- *contamimants in underground water*
- *amorphous solids*
- *convective patterns*
- *polymeric systems*
- *deterministic maps*

Self-Similar Structure of Brownian and Lévy Motions



Normal diffusion

$$\left\langle X^2(t)\right\rangle \propto t$$

Levy Superdiffusion

$$\left\langle X^2(t)\right\rangle \propto t^{\mu},$$

$$\mu = 1.66 > 1$$

x100

x100

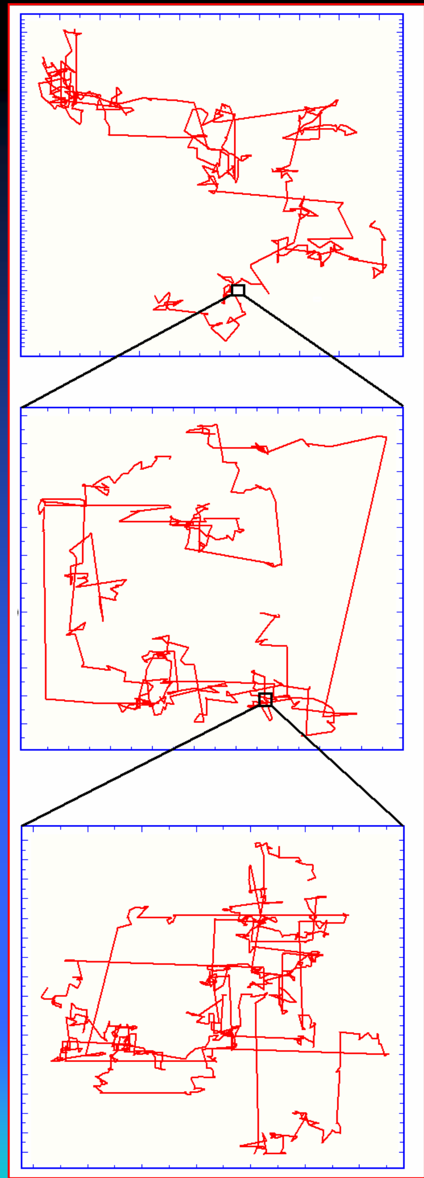
"If, veritably, one took the position from second to second, each of these rectilinear segments would be replaced by a polygonal contour of 30 edges, **each being as complicated as the reproduced design**, and so forth."

J.B. Perren

x100

x100

"The stochastic process which we call linear Brownian motion is a schematic representation which describes well the properties of real Brownian motion, observable on a small enough, but not infinitely small scale, and which assumes that the **same properties exist on whatever scale**." **P.P. Lévy**

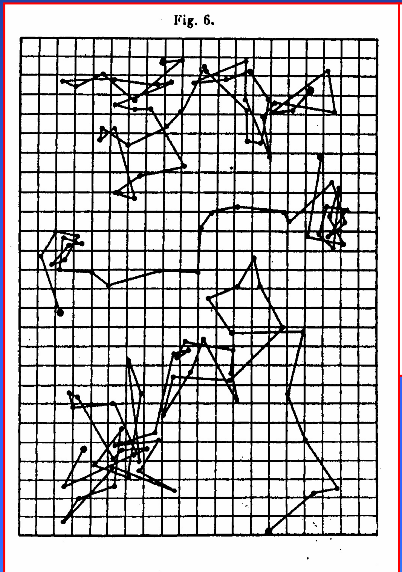


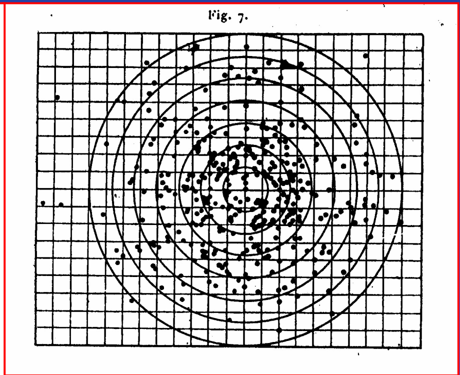
NORMAL vs ANOMALOUS DIFFUSION

Brownian motion of a small grain of putty

(Jean Baptiste Perrin: 1909)

$$\left\langle R^2(t) \right\rangle \propto t$$



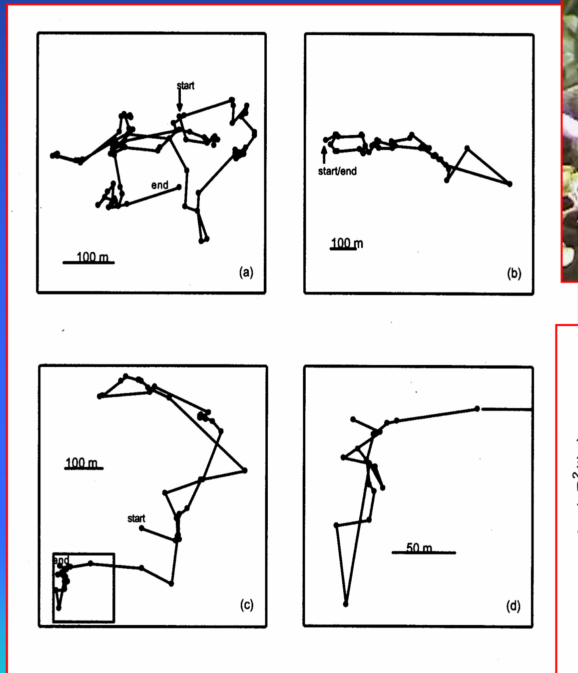


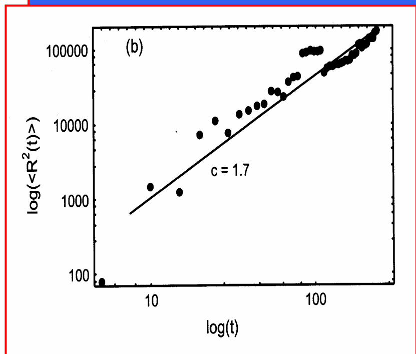
Foraging movement by Lévy flights: optimal strategy to search for food resources distributed at random ?

Gabriel Ramos-Fernandes et al. (2004)

$$\left\langle R^2(t) \right\rangle \propto t^{\mu} \; , \mu \approx 1.7$$

Spider monkeys

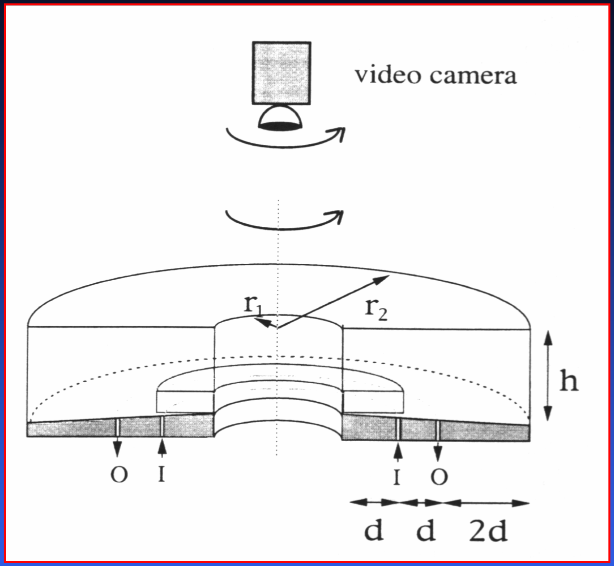


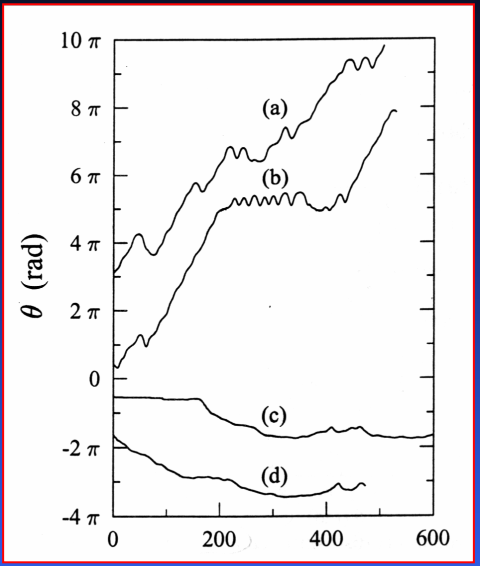


Diffusion of tracers in fluid flows.

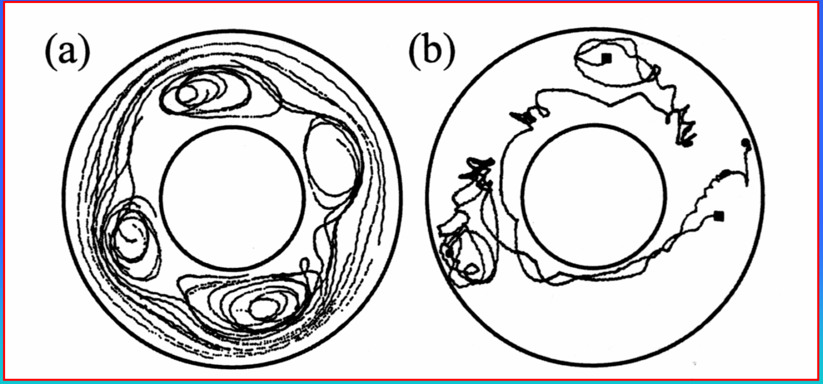
Large scale structures (eddies, jets or convection rolls) dominate the transport.

Example. Experiments in a rapidly rotating annulus (Swinney et al.).



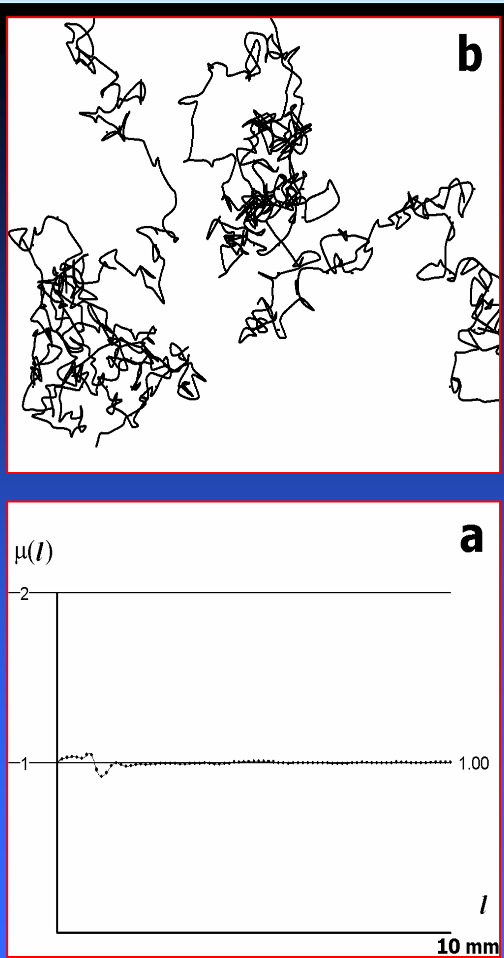


Ordered flow:
Levy diffusion
(flights and traps)
$\mu \approx 1.5 - 1.8$



Weakly turbulent flow:
Gaussian diffusion
$\mu \approx 1$

Anomalous Diffusion in Channeling



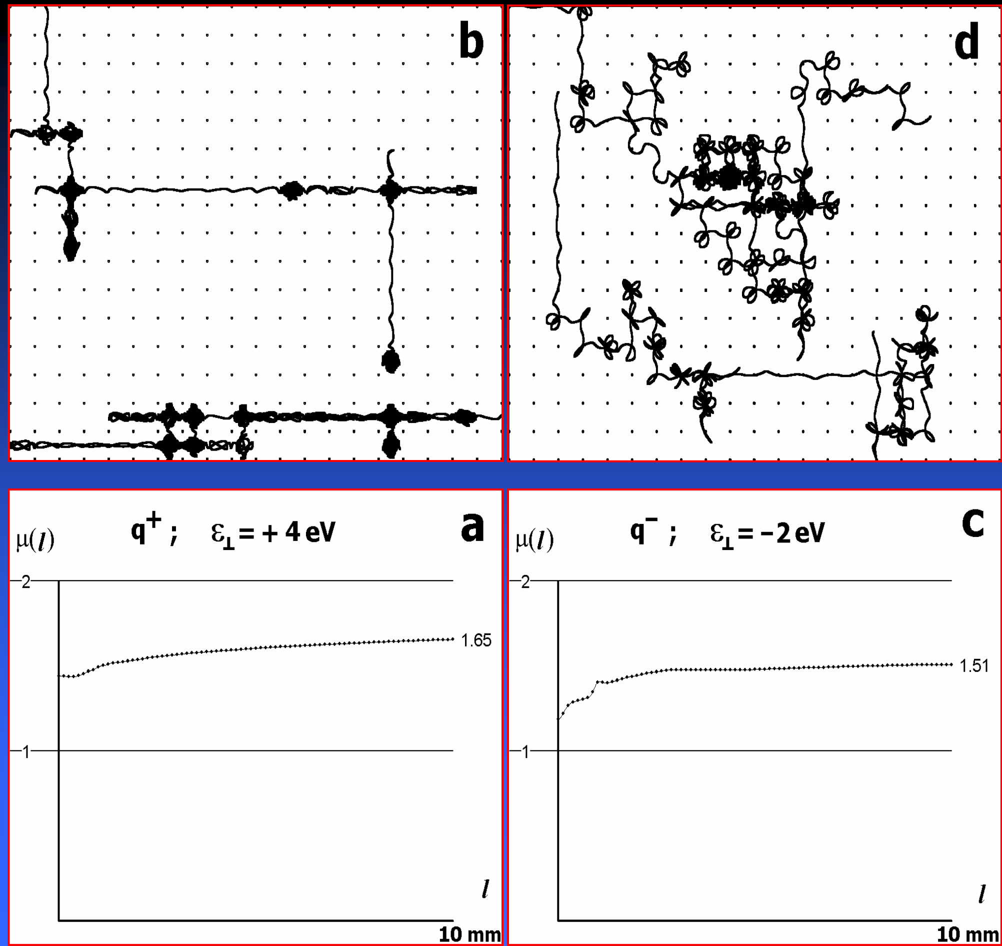
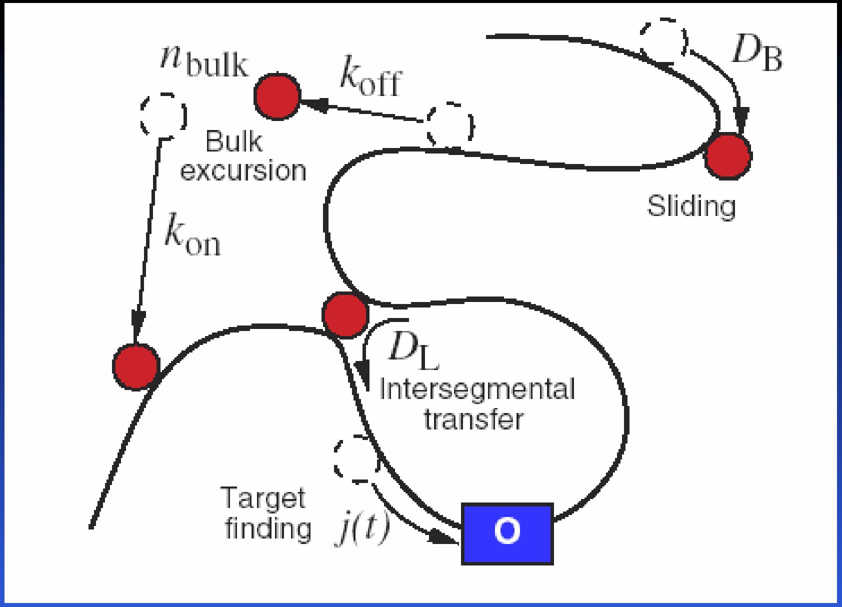


Fig.1. NORMAL DIFFUSION: randomly distributed atom strings

Fig.2. SUPERDIFFUSION: positively and negatively charged particles in a periodic field of atom strings along the axis <100> in a silicon crystal.

"Paradoxical" Diffusion in Chemical Space (Sokolov et al.1997)

Protein diffusion to next neighbor sites on folding DNA (deoxyribonucleic acid)



- Motion of binding proteins or enzymes along DNA: detach to a volume → reattach before reaching the target (Berg – von Hippel model)

- Intersegmental jumps permitted at chain contact points due to polymer looping

- Contoured length |x| stored in a loop between contact points

$$\lambda(x) \sim |x|^{-1-\alpha}$$
$$\alpha = 1/2$$

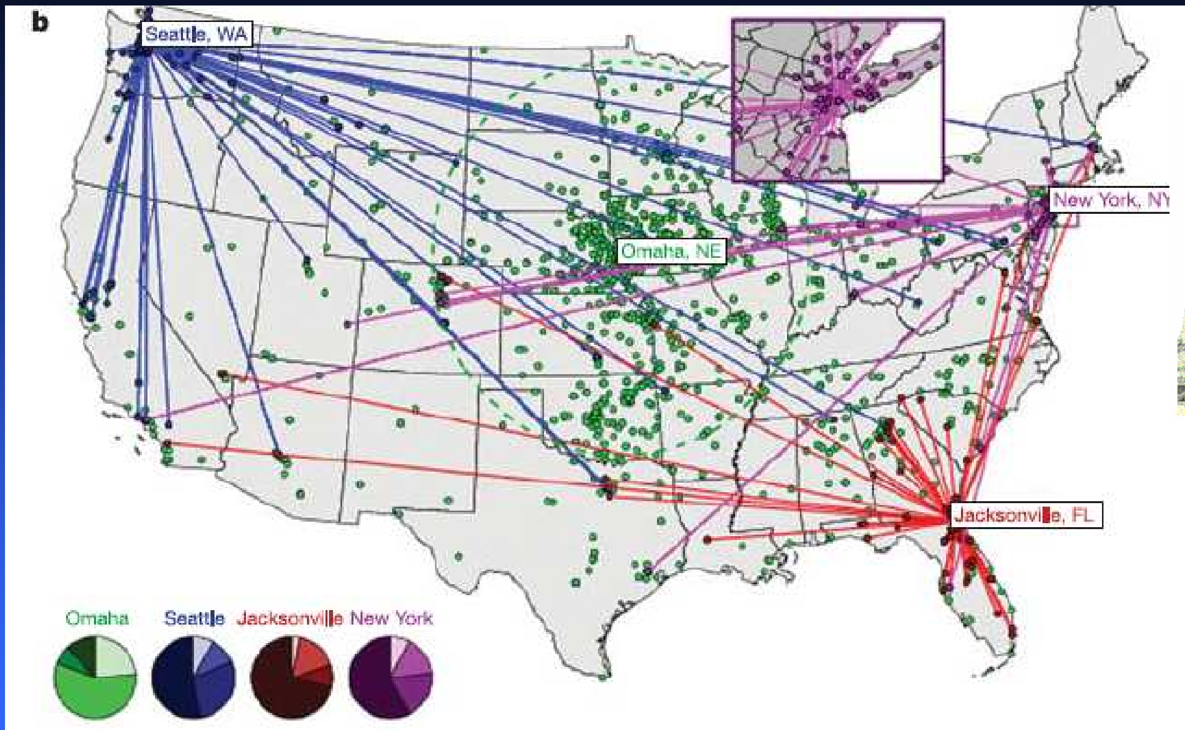
Optimal Target Search on a Fast-Folding Polymer Chain (Lomholt et al.2005)

$$\frac{\partial}{\partial t} n(x,t) = \left(D_B \frac{\partial^2}{\partial x^2} + D_L \frac{\partial^\alpha}{\partial |x|^\alpha} - k_{off} \right) n(x,t) + k_{on} n_{bulk} - j(t)\delta(x)$$

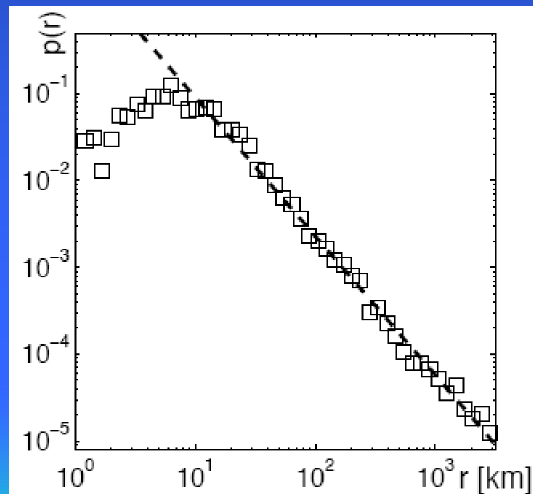
Superdiffusion of bank notes in the US, $\mu \approx 2$

(Brockmann, Hufnagel, Geisel, Nature 2006)

Figure: trajectories of banknotes originated from 4 places : reminiscent of Lévy flights



www.wheresgeorge.com



$$r = \left|\vec{x}_2 - \vec{x}_1\right|$$

Geographical displacement *r* between two report location of a bank note

$$\frac{\partial^{\beta}}{\partial t^{\beta}} W(t,r) = D \frac{\partial^{\alpha}}{\partial |r|^{\alpha}} W(t,r)\,, \;\; \alpha \approx \beta \approx 0.6,\; \mu = 2\beta/\alpha$$

A Lévy Flight for Light

(P. Barthelemy et al, Nature 2008)

- **New optical material in which light performs a Lévy flight**
- **Ideal experimental system to study Lévy flights in a controlled way**
- **Precisely chosen distribution of glass microspheres of different diameters d**

$P(d) \sim d^{-(2 + \alpha)}$

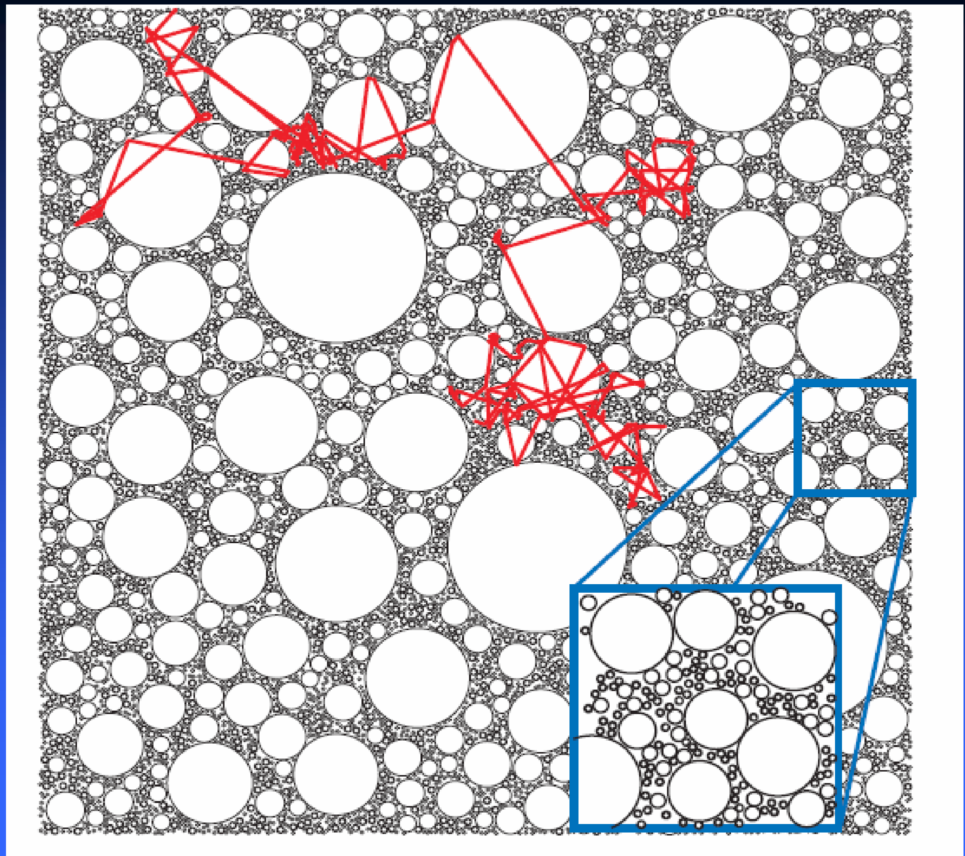


Figure: Lévy walker trajectory in a scale – invariant Levy glass

Lévy flights of photons in hot atomic vapours
(Mercadier et al. Nature Physics 2009)

- Measuring the step size distribution of photons

- Under Gaussian assumptions of emission and absorption spectra (Doppler broadening)

$$P(x) \approx \frac{1}{x^2 \sqrt{\ln(x/l_0)}}$$

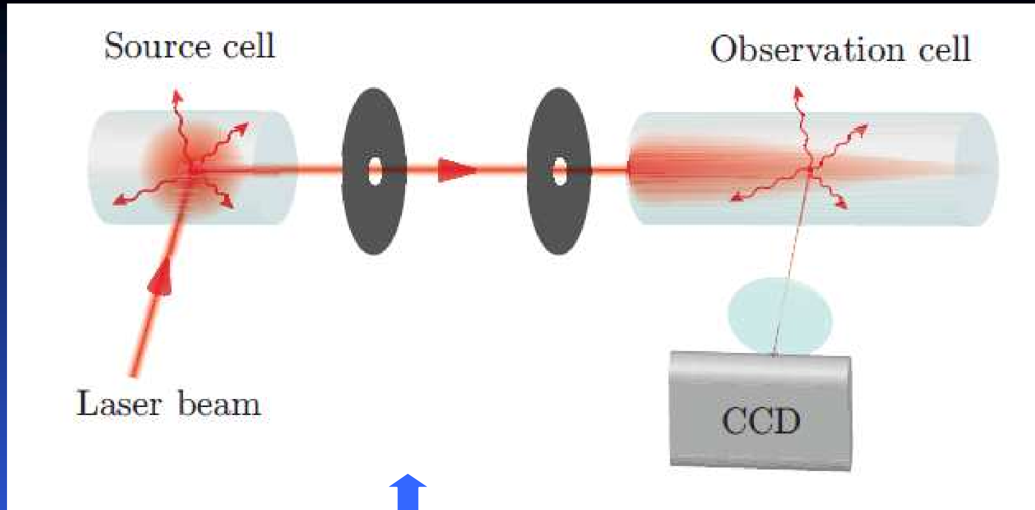


Figure: Double cell configuration to measure the step size PDF

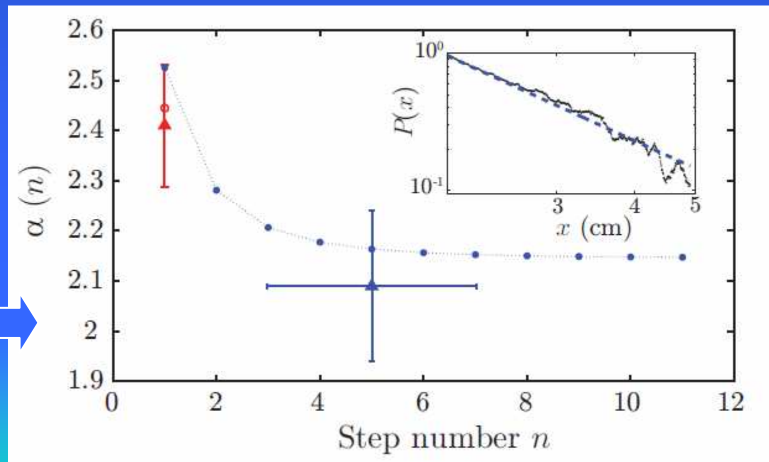


Figure: power law exponent α of the step size distribution vs number *n* of scattering events (multiply scattering)

Impulsive Noises Modeled with the Lévy Stable Distributions

Naturally serve for the description of the processes **with large outliers, far from equilibrium**

Examples include :

- economic stock prices and current exchange rates (1963)
- radio frequency electromagnetic noise
- underwater acoustic noise
- noise in telephone networks
- biomedical signals
- stochastic climate dynamics
- turbulence in the edge plasmas of thermonuclear devices (Uragan 2M, ADITYA, 2003; Heliotron J, 2005 ...)

Lévy noises and Lévy motion

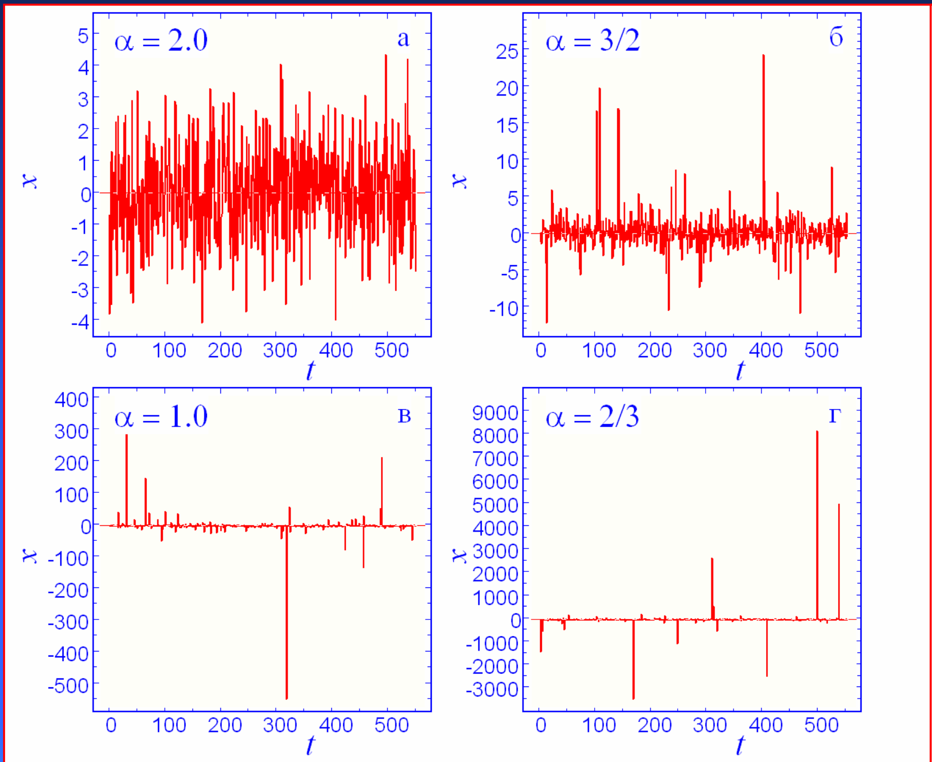
LÉVY NOISES

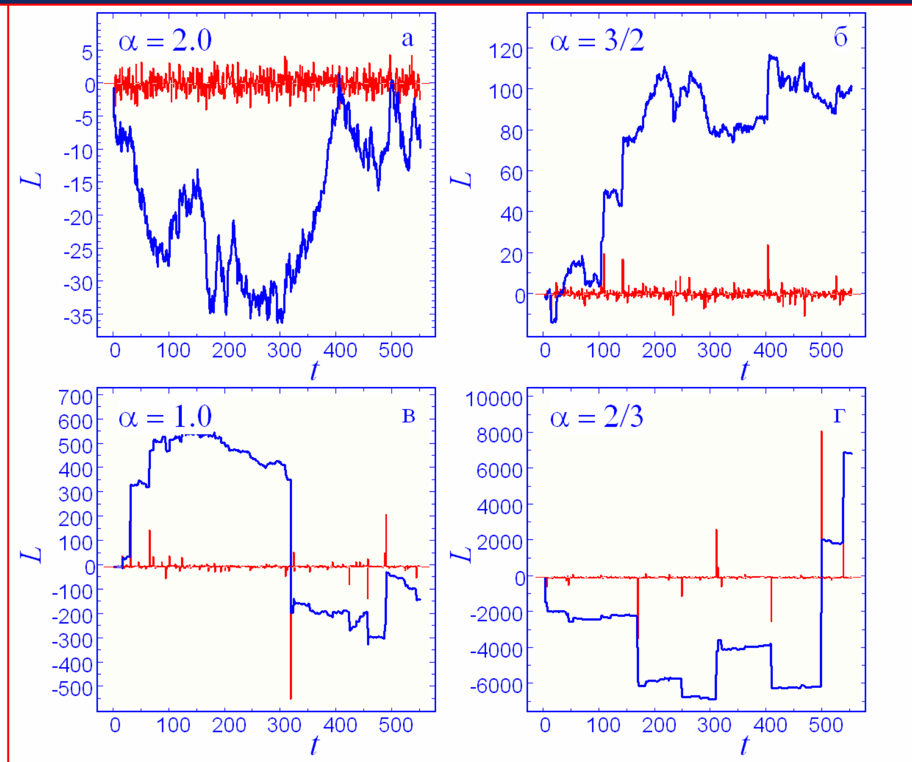
Lévy index ↓ ⇒ outliers ↑

LÉVY MOTION:

successive additions of the noise values

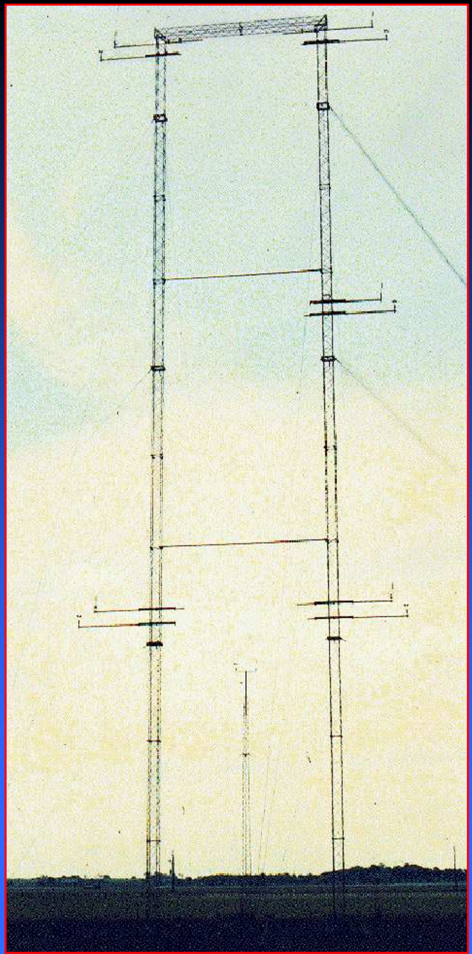
Lévy index ↓ ⇒ “flights” become longer



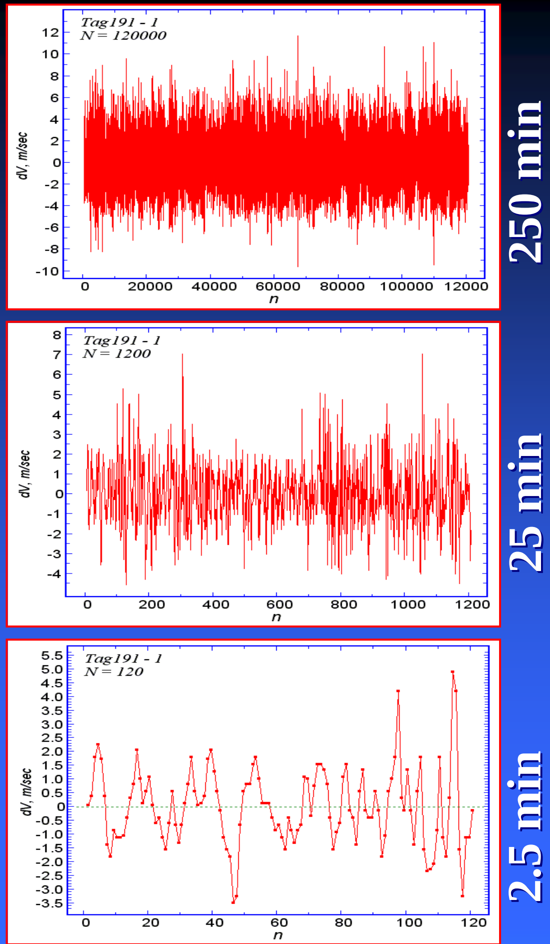


EXPERIMENTAL OBSERVATION OF LÉVY FLIGHTS IN WIND VELOCITY (Lammefjord, denmark, 2006; Gonchar et al.)

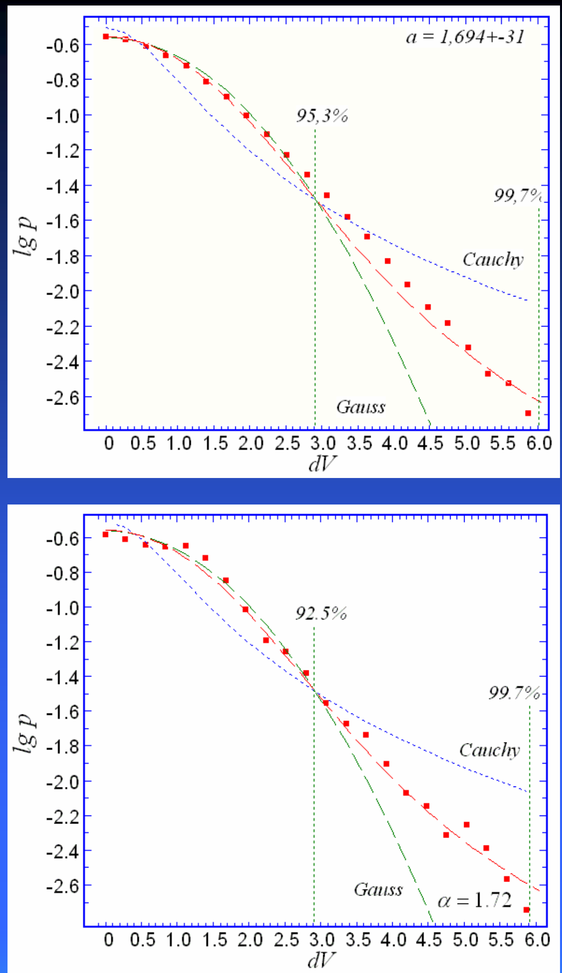
Measuring masts



Velocity increments

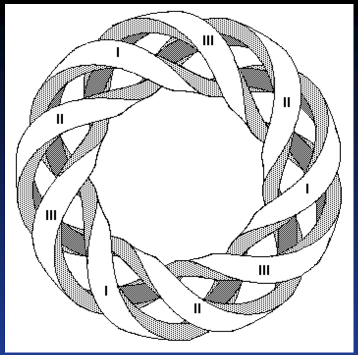


Distribution of increments



Important for reliable prediction of fatique loads

«Lévy Turbulence» in boundary plasma of stellarator «Uragan 3M» (Gonchar et al., 2003)



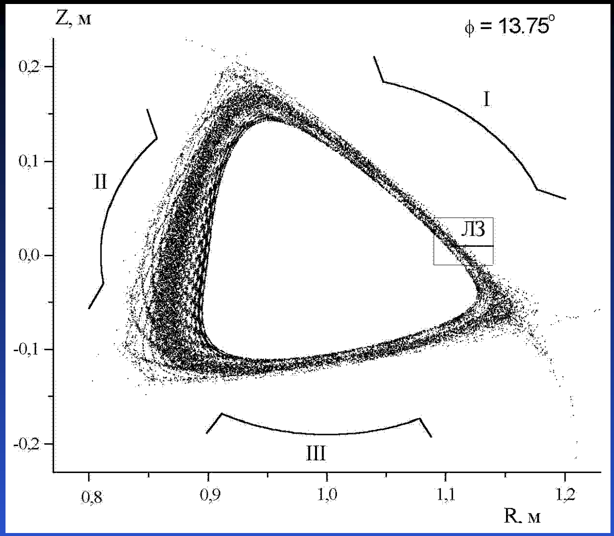
$l = 3, m = 9, R_0 = 1$ m, $\bar{a} \approx 0.1$ m

$B_\varphi = 0.7$ T, $\iota(\bar{a})/2\pi \approx 0.3$,

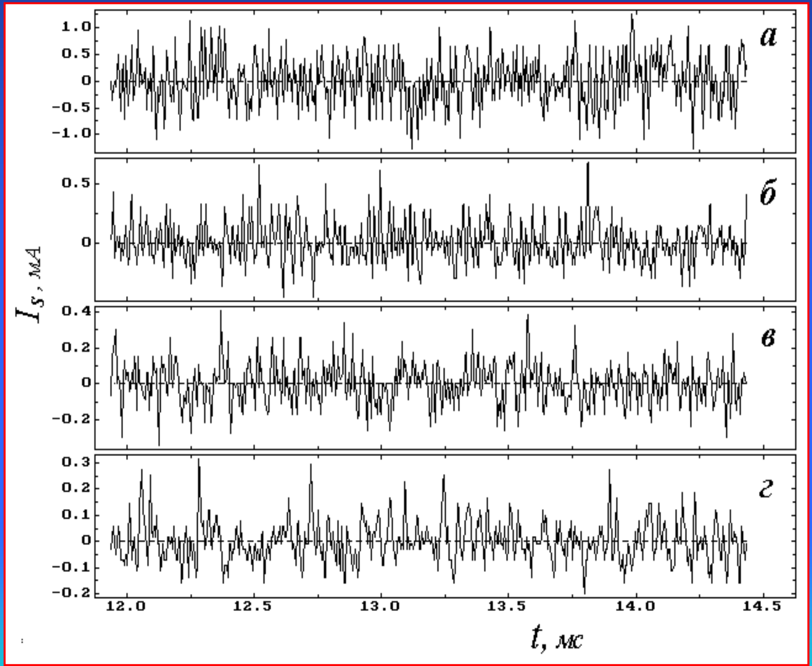
$P_{RF} \approx 200$ kW, $\tau = 60$ ms,

$\bar{n}_e \approx 10^{18}$ m^{-3}, $T_e(0) \approx 500$ eV, $T_i < 100$ eV

Helical magnetic coils (from above)



Poincare section of magnetic line , Л3 – Langmuir prove



Ion saturation current at different probe positions:

а) *R* = 111 cm ; *б*) *R* = 112 cm ;

в) *R* = 112.5 cm ; *г*) *R* = 113 cm.

Boundary region is important for confinement

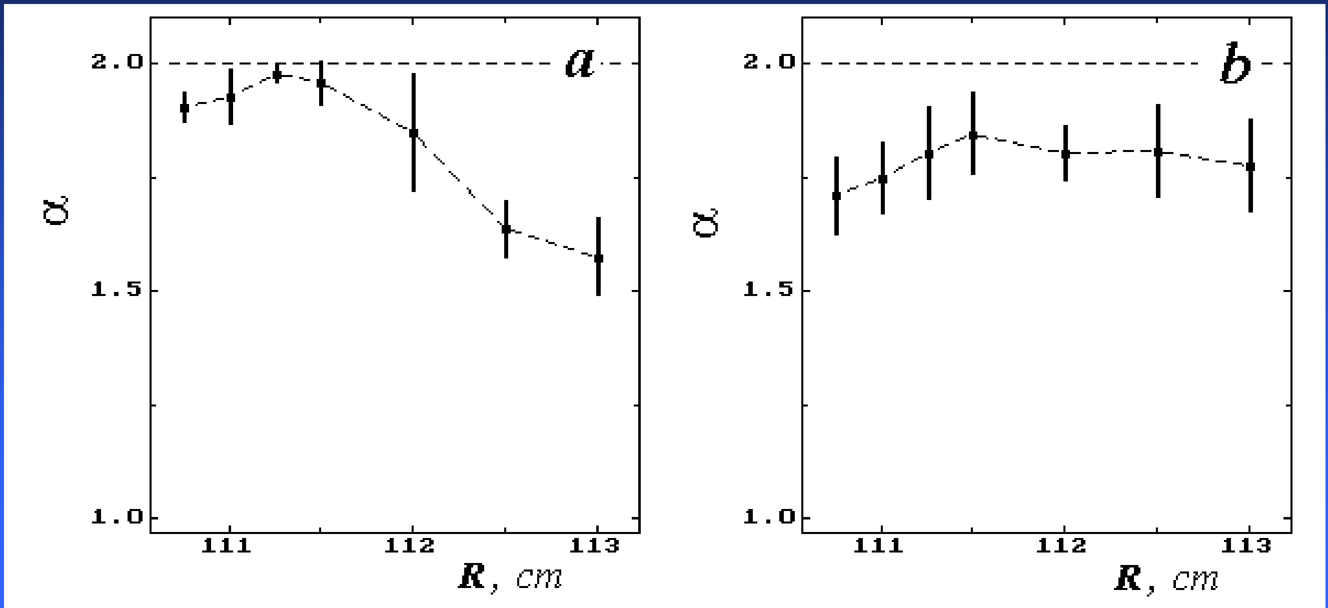
«Lévy Turbulence» in boundary plasma of stellarator «Uragan 3M»

1. Kurtosis and "chi-square" criterium: evidence of non-Gaussianity.

2. Modified method of percentiles: Lévy index of turbulent fluctuations

Ion saturation current ~ δn

Floating potential ~ δφ



• Fluctuations of density and potential measured in boundary plasma of stellarator "Uragan 3M" obey Lévy statistics

- similar conclusions about the Lévy statistics of plasma fluctuations have been drawn for
  - ADITYA
  - L-2M
  - LHD, TJ-II
  - Heliotron J

"bursty" character of fluctuations in other devices (DIII-D, TCABR, …)

"*Levy turbulence*" is a widely spread phenomenon $\Rightarrow$ **Models are strongly needed !**

Two advanced concepts

- **Truncated Levy Flights: PDFs resembles Lévy stable distribution in the central part, however at greater scales the asymptotics decay faster, than the Lévy stable ones,** $\boxed{\langle x^2 \rangle < \infty}$ $\Rightarrow$ **the Central Limit Theorem is applied** $\Rightarrow$ **at large times the PDF tends to Gaussian, however, *sometimes very slowly***

- **Levy Walks: finite velocity when the motion of a massive object is considered**

Might be important for the bumblebees in a finite box ???

Recent reviews

A. Chechkin, V. Gonchar, J. Klafter and R. Metzler, *Fundamentals of Lévy Flight Processes*. **Adv. Chem. Phys**. 133B, 439 (2006).

A. Chechkin, R. Metzler, J. Klafter, V. Gonchar, *Introduction to the Theory of Lévy Flights*. In R. Klages, G. Radons, I.M. Sokolov (Eds), **Anomalous Transport: Foundations and Applications**, **Wiley-VCH**, Weinheim (2008).

R. Metzler, A. Chechkin, J. Klafter, *Lévy statistics and anomalous transport: Lévy flights and subdiffusion*. **Encyclopaedia of Complexity and System Science**. **Springer-Verlag, 2009** (ArXiv:0706.3553v1).

Thank you for attention !