Digraph girth via chromatic number

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Abstract

Let $D$ be a digraph. The chromatic number $\chi(D)$ of $D$ is the smallest number of colours needed to colour the vertices of $D$ such that every colour class induces an acyclic subdigraph. The girth of $D$ is the length of a shortest directed cycle, or $\infty$ if $D$ is acyclic. Let $G(k,n)$ be the maximum possible girth of a digraph on $n$ vertices with $\chi(D) > k$. It is shown that $G(k,n) \geq \lfloor n^{1/k} \rfloor$ and $G(k,n) \leq (3\log_2 n \log_2 \log_2 n)^{1-1/k} n^{1/k}$ for $n \geq 3$ and $k \geq 2$.

1 Introduction

The chromatic number $\chi(D)$ of a digraph $D$ is the minimum number $k$ such that $V(D)$ can be partitioned into $k$ parts, none of which contains a cycle of $D$ (see [2, 10]). By a cycle we always mean a directed cycle, and we define the girth of $D$ as the length of a shortest cycle in $D$ ($\infty$ if $D$ is acyclic).

Given a digraph $D$ with $n$ vertices and chromatic number more than $k$, how large can the girth of $D$ be? This question was posed by one of the authors [9], who conjectured a bound of $O(\sqrt{n})$ in the case $k = 2$. The analogous question for the usual chromatic number in graphs has a long history. A celebrated result of Erdős [5] shows that there are graphs with both girth and chromatic number larger than any specified constant. An example of a quantitative answer to the question is for graphs with girth at least 3 (i.e. triangle-free): the maximum chromatic number of a triangle-free graph on $n$ vertices is $\Theta(\sqrt{n}/\log n)$ by results of Ajtai, Komlós and Szemerédi [1] and of Kim [8]. At the other extreme, a graph on $n$ vertices with chromatic number 3 could consist of single odd cycle of length $n$ or $n – 1$. On the other hand, any graph with chromatic number at least 4 contains a subgraph with minimum degree at least 3, and so a cycle of length $O(\log n)$; probabilistic constructions (see [4]) show that this is the correct order of magnitude. Similar bounds apply for the acyclic chromatic number of a graph $G$, which is the minimum number $k$ such that $V(G)$ can be partitioned into $k$ parts, none of which contains a cycle of $G$. In one direction this is because the
The chromatic number is at least the acyclic chromatic number; for the other, it is not hard to adapt the probabilistic construction to obtain graphs of girth $\Omega(\log n)$ and acyclic chromatic number larger than any fixed constant.

Another motivation to study the aforementioned question became apparent in a recent work of Harutyunyan and Mohar [7]. They generalized to digraphs an old result of Bollobás [3] that for every $k \geq 4$ there is $\alpha > 0$ and infinitely many graphs $G$ of chromatic number $k$ such that every $4$-chromatic subgraph of $G$ contains at least $\alpha|V(G)|$ vertices. The extension to digraphs obtained in [7] proves the same for all $3$-chromatic subdigraphs, but the conclusion does not hold for $2$-chromatic subdigraphs, as every digraph $D$ with $\chi(D) \geq 3$ contains a cycle of length $o(|V(D)|)$, which gives a small $2$-chromatic subdigraph. The last conclusion is a consequence of our Theorem 2 (the case $k = 2$).

## 2 Short cycles in digraphs

Let $G(k, n)$ be the maximum possible girth of a digraph on $n$ vertices with $\chi(D) > k$. Note that the $n$-cycle $C_n$ has $\chi(C_n) = 2$, so $G(1, n) = n$. Thus we may suppose that $k \geq 2$. We start with a lower bound for $G(k, n)$. Note that the order of magnitude is very different than that for graphs.

**Theorem 1.** For every $k \geq 2$ we have $G(k, n) \geq \lfloor n^{1/k} \rfloor$.

**Proof.** Consider the following construction. Let $C_r^i = C_r$ denote the directed cycle of length $r$. For $i \geq 1$ let $C_r^{i+1}$ denote the digraph on $r^{i+1}$ vertices, divided into $r$ parts $V_j$, $j \in \mathbb{Z}_r$ of size $r^i$, so that each part $V_j$ induces a copy of $C_r^i$, and for each $j \in \mathbb{Z}_r$ we have all edges from $V_j$ to $V_{j+1}$. Observe that the girth of $C_r^i$ is equal to $r$. We claim that $\chi(C_r^i) \geq i + 1$ for $1 \leq i \leq r - 1$. This is clear for $i = 1$. Now we argue by induction for $i \geq 2$. Consider any colour class $X$ in any colouring of $C_r^i$. Since $C_r^i[X]$ is acyclic there must be some part $V_j$ disjoint from $X$. Then $C_r^{i-1}[V_j] = C_r^{i-1}$ is coloured using one fewer colour, so $\chi(C_r^i) \geq \chi(C_r^{i-1}) + 1 \geq i + 1$.

To deduce the theorem, let $r = \lfloor n^{1/k} \rfloor$, and let $D$ be the digraph obtained from $C_r^k$ by adding $n - r^k$ isolated vertices. (We can assume $r \geq 3$, as the theorem is obviously true when $r \leq 2$.) Then $\chi(D) > k$ and the girth of $D$ is $r$. □

We remark that $\chi(C_r^i) = i + 1$ for $1 \leq i \leq r - 1$; this is easy to prove by induction.

Our main result is an upper bound that matches the lower bound up to a polylogarithmic factor. Define $g(k, n) = (3 \log_2 n \log_2 \log_2 n)^{1-1/k} n^{1/k}$.

**Theorem 2.** For $n \geq 3$ and $k \geq 2$ we have $G(k, n) \leq g(k, n)$.

To prove Theorem 2 we introduce an additional digraph parameter. We say that $S \subseteq V(D)$ is a hitting set if every cycle in $D$ contains at least one vertex of $S$. Let $H(r, n)$ be the smallest number $h$ such that any digraph $D$ on $n$ vertices with girth more than $r$ has a hitting set of size $h$. Note that if $r \geq n$ such a digraph is acyclic, so $H(r, n) = 0$. Let $h(r, n) = 3(n/r) \log_2 n \log_2 \log_2 n$.

**Theorem 3.** For $n \geq r \geq 3$ we have $H(r, n) \leq h(r, n)$. 

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Proof. For every fixed $r \geq 3$, we argue by induction on $n$. The base case is $n = r$, when $H(r, n) = 0$. Now suppose that $n > r$. Note that we can assume that $h(r, n) < n$, since the entire vertex set is trivially a hitting set, so we have $r > 3 \log_2 n \log_2 \log_2 n$. Since $3 \log_2 n \log_2 \log_2 n > n$ for $3 \leq n \leq 37$ we can assume that $r \geq 38$. The idea for the induction step is as follows. Suppose $D$ is a digraph on $n$ vertices with no cycle of length at most $r$. We find a small set $S$ of vertices and a partition of $V(D) \setminus S$ as $A \cup B$ so that there are no edges of $D$ from $A$ to $B$. Then we apply the induction hypothesis to find hitting sets in $D[A]$ and $D[B]$, to which we add $S$ to obtain a hitting set in $D$.

To find $S$ we fix any vertex $v$ and consider its iterated neighbourhoods, defined as follows. Given a vertex $u$, the *out-distance* of $u$ from $v$ is the length of a shortest path in $D$ from $v$ to $u$ (or $\infty$ if there is no such path). The *in-distance* of $u$ from $v$ is the out-distance of $v$ from $u$. Let $N_1^+(v)$ be the set of vertices at out-distance $1$ from $v$ and $N_1^-(v)$ be the set of vertices at in-distance $1$ from $v$. Let $N_{\leq i}(v) = \cup_{j=1}^i N_{j}^+(v)$ and $N_{\geq i}(v) = \cup_{j=1}^i N_{j}^-(v)$. Let $t = \lfloor r/2 \rfloor$. Note that $N_{\leq t}(v) \cap N_{\geq t}(v) = \emptyset$, since there is no cycle of length at most $r$.

Now we suppose for a contradiction that $D$ does not have a hitting set of size $h(r, n)$. We will see that this forces the iterated neighbourhoods of $v$ to grow rapidly (see [6] for a similar argument based on edge expansion). To see this, fix $i < t$, let $S_i^+ = N_{i+1}^+(v)$, $A_i^+ = N_{\leq i}^+(v)$, $B_i^+ = V(D) \setminus (A_i^+ \cup S_i^+)$, and note that there are no edges of $D$ from $A_i^+$ to $B_i^+$. Write $m = |A_i^+|$. By induction hypothesis, $D[A_i^+]$ has a hitting set of size $h(r, m)$ and $D[B_i^+]$ has a hitting set of size $h(r, |B_i^+|) \leq h(r, n - m)$. Adding $S_i^+$ gives a hitting set of $D$, which by assumption has size more than $h(r, n)$, so $|S_i^+| > h(r, n) - h(r, m) - h(r, n - m)$. We estimate $h(r, n) - h(r, m) - h(r, n - m) \geq 3r^{-1} \log_2 n \log_2 n - m \log_2 m - (n - m) \log_2 n \geq c_i^+ |A_i^+|$, where $c_i^+ = 3r^{-1} \log_2 \frac{n}{|A_i^+|} \log_2 n$. Therefore $|A_{i+1}^+| = |A_i^+| + |S_i^+| > (1 + c_i^+) |A_i^+|$.

To estimate the growth of $|A_i^+|$ we divide the steps into groups $G_j$, $j \geq 1$, such that for $i \in G_j$ we have $n^{1 - 2^{-j+1}} \leq |A_i^+| < n^{1 - 2^{-j}}$. Then for each $i \in G_j$ we have $|A_{i+1}^+|/|A_i^+| > 1 + d_j$, where $d_j := 3r^{-2} \log_2 n \log_2 \log_2 n$. Also, the total expansion factor over $i \in G_j$, excluding the last element of $G_j$, is at most $n^{2^{-j}}$. Therefore $(1 + d_j)^{|G_j| - 1} \leq n^{2^{-j}}$. Note that $d_j < 1$, as $r > 3 \log_2 n \log_2 \log_2 n$. Using the inequality $(1 + 1/x)^x \geq 2$ for $x \geq 1$ we obtain $n^{2^{-j}} \geq 2^{d_j (|G_j| - 1)}$, so $|G_j| - 1 \leq d_j^{-1} 2^{-j} \log_2 n = \frac{r}{3 \log_2 \log_2 n}$. Let $\ell \in \mathbb{N}$ be such that $n^{1 - 2^{-\ell+1}} \leq n/2 < n^{1 - 2^{-\ell}}$; then $\log_2 \log_2 n < \ell \leq 1 + \log_2 \log_2 n$. Thus we reach a set $|A_{i^+}^+| > n/2$ for some $i^+$, where

\[
i^+ \leq \sum_{j=1}^{\ell} |G_j| \leq (1 + \lfloor \log_2 \log_2 n \rfloor) \left(1 + \frac{r}{3 \log_2 \log_2 n}\right).
\]

Since $n > r \geq 38$, we have $\log_2 \log_2 n \geq 2$. If $\lfloor \log_2 \log_2 n \rfloor = 2$, then (1) implies that

\[
i^+ \leq 3 + \frac{r}{\log_2 \log_2 n} \leq 3r^{38} + \frac{r}{\log_2 \log_2 39} \leq \frac{r}{2}.
\]

On the other hand, if $\log_2 \log_2 n \geq 3$, then $r > 3 \log_2 n \log_2 \log_2 n \geq 72$. In this case, (1) gives the
same conclusion as above:

\[ i^+ \leq (1 + \log_2 \log_2 n) \left(1 + \frac{r}{3 \log_2 \log_2 n}\right) \]

\[ \leq \frac{r}{3} + 1 + \frac{r}{3 \log_2 \log_2 n} + \frac{r}{3 \log_2 n} \]

\[ \leq \frac{r}{3} + \frac{r}{72} + \frac{r}{9} + \frac{r}{24} = \frac{r}{2}. \]

In the last calculation we used \( r > 3 \log_2 n \log_2 \log_2 n \) and \( r \geq 72. \)

The same argument applies to \( A_i^- = N^-_{i-1}(v), \) so we reach a set \(|A_i^-| > n/2\) for some \( i^- \leq r/2. \)

But then \( A_i^+ \) and \( A_i^- \) intersect, contradicting the assumption that there is no cycle of length at most \( r. \) Thus \( D \) does have a hitting set of size \( h(r, n), \) which completes the proof by induction. \( \square \)

Proof of Theorem 2. Suppose that \( D \) is a digraph on \( n \) vertices with girth more than \( r = g(k, n). \)

We claim that \( D \) has chromatic number at most \( k. \) To see this, we repeatedly apply Theorem 3 (we can assume \( n \geq r \geq 3). \)

Let \( S_1 = V(D). \) For \( i \geq 2 \) we apply Theorem 3 to find a hitting set \( S_i \) for \( D[S_{i-1}]. \) Then \( |S_i| \leq h(r, |S_{i-1}|) \leq (3r^{-1} \log_2 n \log_2 \log_2 n)^{i-1} n \) and \( D[S_{i-1} \setminus S_i] \) is acyclic for \( i \geq 2. \)

Since \( |S_k| \leq (3r^{-1} \log_2 n \log_2 \log_2 n)^{k-1} n \leq r \) and \( D \) has girth more than \( r, \) \( D[S_k] \) is acyclic.

Thus we have a \( k \)-colouring, whose colour classes are \( S_{i-1} \setminus S_i (2 \leq i \leq k) \) and \( S_k. \)

\( \square \)

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References


