MTH4103: Geometry I

Queen Mary
University of London

Mid-term Test 2014 Solutions

Duration: 40 minutes

Date and Time: 19th February 2014, in the hour 11:00–12:00

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The duration of this test is **40 minutes**. Answer **all** the questions. The marks (out of 100) allocated to each question are shown after the question numbers. For most questions, only the final answer to a question will be marked, so indicate this answer clearly. Simplify your answers as much as possible.

Calculators are **not** permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Answer all questions in the spaces provided. You may do additional rough work on the backs of the question sheets or at the end, but this will not be looked at.

Do **not** turn over until an invigilator instructs you to do so.

For Examiner's use only

Question	Mark	Question	Mark
1	10	5	20
2	10	6	20
3	10	7	10
4	10	8	10
		Total (%)	100

Question 1 (10 marks). Let $\mathbf{a} = \begin{pmatrix} 4 \\ -7 \\ -4 \end{pmatrix}$. Determine $|\mathbf{a}|$.

Solution. We have $|\mathbf{a}| = \sqrt{4^2 + (-7)^2 + (-4)^2} = \sqrt{16 + 49 + 16} = \sqrt{81} = 9.$✓

Question 2 (10 marks). Determine parametric equations for the line ℓ through the points $A = (-1, -4, 3)$ and $B = (0, -1, 2)$.

Solution. Let $A = (-1, -4, 3)$ and $B = (0, -1, 2)$ have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Then the line ℓ through A and B has direction

$$\mathbf{u} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix},$$

and so has parametric equations

$$\left.\begin{array}{l} x = -1 + \lambda \\ y = -4 + 3\lambda \\ z = 3 - \lambda \end{array}\right\}.$$ ✓

Question 3 (10 marks). Consider the following system of linear equations defined over $\mathbb{R}$ in variables x, y and z:

$$\left.\begin{aligned} 3y + 2z &= -1 \\ -3x - 2y + 3z &= -2 \\ x + 4y + 9z &= -4 \end{aligned}\right\}.$$

State clearly, and in order, **all** operations that must be performed in Step 1 of the Gaußian elimination process (as defined in this module) to bring the above system of equations into echelon form. [You do not have to perform these operations, except if you have trouble visualising the operations you have to do. Nor will the system of equations be in echelon form at the end of Step 1.]

Solution. The (two) operations are as follows.

1. Interchange (swap) Equations 1 and 2.
2. Add $\frac{1}{3}$ times Equation 1 to Equation 3.✓

Question 4 (10 marks). Define carefully, but **without using coördinates**, the *scalar product* (or *dot product*) $\mathbf{u}\cdot\mathbf{v}$ of two vectors $\mathbf{u}$ and $\mathbf{v}$.

Solution. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ then we define $\mathbf{u}\cdot\mathbf{v} := 0$. Else $\mathbf{u}\cdot\mathbf{v} := |\mathbf{u}||\mathbf{v}|\cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.✓

Question 5 (20 marks).

(a) **(7 marks)** Given two planes in $\mathbb{R}^3$, describe (geometrically) their possible intersections. Also say which (case or) cases are typical, and which are exceptional.

(b) **(3 marks)** Given three planes in $\mathbb{R}^3$, what do we expect their intersection to be in general? Ignore exceptional cases.

(c) **(10 marks)** Planes Π_1 and Π_2, with Cartesian equations $3x + y - z = 3$ and $2y - z = 4$ respectively, intersect in a line ℓ. Find a vector equation for ℓ.

Solution.

(a) The possible cases are as follows.

  (i) A line (typical).
  (ii) The empty set (atypical).
  (iii) A plane ([very] atypical).✓

(b) A point.✓

(c) We observe that the system of equations

$$\left.\begin{aligned} 3x + y - z &= 3 \\ 2y - z &= 4 \end{aligned}\right\}.$$

specifying the intersection of Π_1 and Π_2 is already in echelon form. Solving by back substitution gives us $z = t$, where t can be any real number. So now $2y = 4 + z$, giving us $y = 2 + (\frac{1}{2})t$, and finally $3x = 3 + z - y = 3 + t - 2 - (\frac{1}{2})t$, which gives us $x = \frac{1}{3} + (\frac{1}{6})t$. So possible vector equations for ℓ include

$$\mathbf{r} = \begin{pmatrix} 1/3 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1/6 \\ 1/2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} 1/3 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}.\checkmark$$

Question 6 (20 marks). A regular tetrahedron in $\mathbb{R}^3$ can be taken to have vertices at $O = (0,0,0)$, $A = (0,1,1)$, $B = (1,0,1)$ and $C = (1,1,0)$, which have position vectors $\mathbf{0}$, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively.

(a) **(7 marks)** Calculate the vector product $\mathbf{a} \times \mathbf{b}$.

(b) **(7 marks)** Calculate the volume of the tetrahedron (this is $\frac{1}{6}\times$ the volume of a parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$).

(c) **(6 marks)** Calculate $\cos\theta$, where θ is the angle between $\mathbf{c} - 2\mathbf{a}$ and the vector represented by $\overrightarrow{AB}$.

Solution.

(a) $\mathbf{a}\times\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\mathbf{k} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.\checkmark$

(b) The volume is $\frac{1}{6}|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| = \frac{1}{6}|\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})| = \frac{1}{6}|1+1+0| = \frac{1}{3}.\checkmark$

(c) We have $\mathbf{c} - 2\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ and $\mathbf{b} - \mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, where $\mathbf{b} - \mathbf{a}$ is the vector represented by $\overrightarrow{AB}$. So $|\mathbf{c} - 2\mathbf{a}| = \sqrt{6}$, $|\mathbf{b} - \mathbf{a}| = \sqrt{2}$ and $(\mathbf{c} - 2\mathbf{a})\cdot(\mathbf{b} - \mathbf{a}) = 1 + 1 + 0 = 2$. Therefore $\cos\theta = \frac{2}{\sqrt{6}\sqrt{2}} = \frac{1}{\sqrt{3}}.\checkmark$

Question 7 (10 marks). Exactly which of the following statements are true, and exactly which are false? [A statement of the form "s_1 and s_2 are true" will *not* be taken to mean that you think s_3, s_4 and s_5 are false.]

(a) If $\mathbf{w}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$ then $\mathbf{w}$ and $\mathbf{u} \times \mathbf{v}$ are collinear.

(b) If $\mathbf{u} \cdot \mathbf{v} = 0$ for all vectors $\mathbf{v}$ then $\mathbf{u} = \mathbf{0}$.

(c) If $\mathbf{u} \cdot \mathbf{v} = 0$ then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

(d) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) - (\mathbf{w} \times \mathbf{u})$ for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

(e) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ for all nonzero vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

Solution. (a), (b) and (d) are true, while (c) and (e) are false.✓

Question 8 (10 marks). Pick a plane Π passing through the points $A = (1, 2, 3)$ and $B = (4, 5, 6)$ with position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. (There are infinitely many such planes to choose from.) Write down a Cartesian equation for Π.

Solution. (NB: Answer not unique.) A specific answer was required, and one possible answer is $2x - y - z = -3$.✓ (The most general possibility for Π has a Cartesian equation $ax + by + cz = c - a$ for some $a, b, c \in \mathbb{R}$ such that $(a, b, c) \neq (0, 0, 0)$ and $a + b + c = 0$.)