MTH4103: Geometry I

Queen Mary
University of London

Mid-term Test 2013 Solutions

Duration: 40 minutes

Date and Time: 22nd February 2013, in the hour 13:00–14:00

Last name: BRAY

First name(s): John Nicholas

Student number: 244823040

The duration of this test is **40 minutes**. Answer **all** the questions. The marks (out of 100) allocated to each question are shown after the question numbers. For most questions, only the final answer to a question will be marked, so indicate this answer clearly. Simplify your answers as much as possible.

Calculators are **not** permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Answer all questions in the spaces provided. You may do additional rough work on the backs of the question sheets or at the end, but this will not be looked at.

Do **not** turn over until an invigilator instructs you to do so.

For Examiner's use only

Question	Mark	Question	Mark
1	10	5	20
2	10	6	20
3	10	7	10
4	10	8	10
		Total (%)	100

Question 1 (10 marks). Let $\mathbf{a} = \begin{pmatrix} -4 \\ 1 \\ -8 \end{pmatrix}$. Determine $|\mathbf{a}|$.

Solution. We have $|\mathbf{a}| = \sqrt{(-4)^2 + 1^2 + (-8)^2} = \sqrt{16 + 1 + 64} = \sqrt{81} = 9.$✓

Question 2 (10 marks). Determine parametric equations for the line ℓ through the points $A = (-2, 1, 0)$ and $B = (4, 3, -2)$.

Solution. (NB: Answer not unique.) Let $A = (-2, 1, 0)$ and $B = (4, 3, -2)$ have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Then the line ℓ through A and B has direction

$$\mathbf{u} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -2 \end{pmatrix},$$

and so has parametric equations

$$\left.\begin{aligned} x &= -2 + 6\lambda \\ y &= 1 + 2\lambda \\ z &= -2\lambda \end{aligned}\right\}.$$✓

Question 3 (10 marks). Let $A = (-1, 1, 2)$ and $B = (-2, 3, 4)$, and let ℓ be the unique line on which A and B lie. Let P be the point on ℓ that is *not* between A and B and such that the distance from P to B is half the distance from A to B. (Thus $|\overrightarrow{AP}| = \frac{3}{2}|\overrightarrow{AB}|$.) Determine the position vector $\mathbf{p}$ of P. [For notational convenience, A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively.]

Solution. The formula is $\mathbf{p} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$, where here $\lambda = \frac{3}{2}$. Thus we get that

$$\mathbf{p} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \frac{3}{2}\left[\begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}\right] = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -\frac{3}{2} \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{5}{2} \\ 4 \\ 5 \end{pmatrix}. \checkmark$$

Question 4 (10 marks). Define carefully, but **without using coördinates**, the *scalar product* (or *dot product*) $\mathbf{u}\cdot\mathbf{v}$ of two vectors $\mathbf{u}$ and $\mathbf{v}$.

Solution. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ then we define $\mathbf{u}\cdot\mathbf{v} := 0$. Else $\mathbf{u}\cdot\mathbf{v} := |\mathbf{u}||\mathbf{v}|\cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. ✓

Question 5 (20 marks). Apply Gaußian elimination to transform the following system of linear equations into echelon form (indicate this clearly), followed by back substitution to determine their full solution set. [You may assume that these are linear equations over $\mathbb{R}$ in the variables x, y and z.]

$$\left.\begin{aligned} y + 2z &= -1 \\ x - 2y + 3z &= -3 \\ -3x + 6y + 9z &= -9 \end{aligned}\right\}.$$

Solution. First, we swap Equations 1 and 2 to get

$$\left.\begin{aligned} x - 2y + 3z &= -3 \\ y + 2z &= -1 \\ -3x + 6y + 9z &= -9 \end{aligned}\right\}.$$

Next, we add 3 times Equation 1 to Equation 3 to get

$$\left.\begin{aligned} x - 2y + 3z &= -3 \\ y + 2z &= -1 \\ 18z &= -18 \end{aligned}\right\},$$

which is now in echelon form. For the back substitution, we observe that all the variables are leading in one of the equations. From $18z = -18$, we get $z = -1$. Then $y + 2z = -1$ gives $y = -1 - 2(-1) = 1$, and then $x - 2y + 3z = -3$ gives $x = 2y - 3z - 3 = 2$. So their solution set is $\{(2, 1, -1)\}$.✓ (This is a point in $\mathbb{R}^3$.)

Question 6 (20 marks). The four vertices of a regular tetrahedron of side length 2 can be taken to have position vectors $\mathbf{0}$, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, where $\mathbf{a} = 2\mathbf{i}$, $\mathbf{b} = \mathbf{i} + \sqrt{3}\mathbf{j}$ and $\mathbf{c}$ is dealt with below.

(a) **(5 marks)** Calculate the vector product $\mathbf{a} \times \mathbf{b}$.

(b) **(5 marks)** Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors of length 2 at an angle of $\frac{\pi}{3}$ to each other. Calculate $\mathbf{u}\cdot\mathbf{v}$. [Hint: $\mathbf{a}$ and $\mathbf{b}$ are two such vectors.]

(c) **(10 marks)** Given that $\mathbf{c}$ has length 2 and has angle $\frac{\pi}{3}$ with both $\mathbf{a}$ and $\mathbf{b}$, determine the two possibilities for $\mathbf{c}$.

Solution.

(a) $\mathbf{a} \times \mathbf{b} = (2\mathbf{i}) \times (\mathbf{i} + \sqrt{3}\mathbf{j}) = 2(\mathbf{i} \times \mathbf{i}) + 2\sqrt{3}(\mathbf{i} + \mathbf{j}) = \mathbf{0} + 2\sqrt{3}\mathbf{k} = 2\sqrt{3}\mathbf{k}.$✓

(b) We have $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\frac{\pi}{3} = 2.2.\frac{1}{2} = 2.$✓

(c) Let $\mathbf{c} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. From $\mathbf{a}\cdot\mathbf{c} = 2$ get $2a = 1$, so that $a = 1$. Now $\mathbf{b}\cdot\mathbf{c} = 2$ gives $1 + \sqrt{3}b = 2$, so that $b = \frac{1}{\sqrt{3}}$. Now $|\mathbf{c}| = 2$ gives

$$4 = |\mathbf{c}|^2 = a^2 + b^2 + c^2 = 1 + \frac{1}{3} + c^2,$$

and so $c^2 = \frac{8}{3}$, and thus $c = \pm\frac{2\sqrt{2}}{\sqrt{3}}$. Therefore $\mathbf{c} = \mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} \pm \frac{2\sqrt{2}}{\sqrt{3}}\mathbf{k}$.✓

Question 7 (10 marks). Exactly which of the following statements are true, and exactly which are false? [A statement of the form "s_1 and s_2 are true" will *not* be taken to mean that you think s_3, s_4 and s_5 are false.]

(a) If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are vectors such that $\mathbf{u}$ is orthogonal to $\mathbf{v}$ and $\mathbf{v}$ is orthogonal to $\mathbf{w}$, then $\mathbf{u}$ is orthogonal to $\mathbf{w}$.

(b) If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are coplanar then $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} \times \mathbf{v}$ are coplanar.

(c) $\mathbf{v} \times \mathbf{u} = \mathbf{u} \times \mathbf{v}$ for all vectors $\mathbf{u}$ and $\mathbf{v}$.

(d) $\mathbf{v} \times \mathbf{u} \neq \mathbf{u} \times \mathbf{v}$ for all vectors $\mathbf{u}$ and $\mathbf{v}$.

(e) $\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w}) = \mathbf{w}\cdot(\mathbf{v} \times \mathbf{u})$ for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

Solution. They are all false.✓

Question 8 (10 marks). Let Π be the plane (in $\mathbb{R}^3$) with the equation $2x+y-2z = 4$. Determine a point (called P) on Π and a nonzero vector (called $\mathbf{n}$) orthogonal to Π. Use this information to determine a point (called Q) at distance 6 from Π. [You may specify $\mathbf{p}$ and $\mathbf{q}$ instead of P and Q, where $\mathbf{p}$ and $\mathbf{q}$ are (respectively) the position vectors of P and Q.]

Solution. (NB: Answer not unique.) One reads off the coëfficients of x, y, z in the equation of Π to get:

$$\mathbf{n} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}.$$

We can take $P = (2, 0, 0)$ (substitute $y = z = 0$ in the equation $2x + y - 2z = 4$).

Now $|\mathbf{n}| = 3$, so we add $2\mathbf{n}$ to $\mathbf{p}$ in order to obtain $\mathbf{q}$. This would give $Q = (6, 2, -4)$ for the case $P = (2, 0, 0)$.✓ (We can also add $-2\mathbf{n}$ to $\mathbf{p}$ to get $\mathbf{q}$. Correct answers must have $\mathbf{q}\cdot\mathbf{n} = 22$ or -14, with $\mathbf{n}$ as above.)