MTH4103: Geometry I

Mid-term Test 2013 Solutions

Duration: 40 minutes

Date and Time: 22nd February 2013, in the hour 13:00–14:00

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The duration of this test is **40 minutes**. Answer **all** the questions. The marks (out of 100) allocated to each question are shown after the question numbers. For most questions, only the final answer to a question will be marked, so indicate this answer clearly. Simplify your answers as much as possible.

Calculators are **not** permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Answer all questions in the spaces provided. You may do additional rough work on the backs of the question sheets or at the end, but this will not be looked at.

Do **not** turn over until an invigilator instructs you to do so.

For Examiner's use only

Question	Mark	Question	Mark
1	10	5	20
2	10	6	20
3	10	7	10
4	10	8	10
		Total (%)	100

Question 1 (10 marks). Let $\mathbf{a} = \begin{pmatrix} -4 \\ 1 \\ -8 \end{pmatrix}$. Determine $|\mathbf{a}|$.

Solution. We have $|\mathbf{a}| = \sqrt{(-4)^2 + 1^2 + (-8)^2} = \sqrt{16+1+64} = \sqrt{81} = 9.$✓

The marking scheme was as follows: 10 marks for the answer "9"; 5 marks for "$\sqrt{81}$"; 0 marks for any other answer, including "± 9".

Question 2 (10 marks). Determine parametric equations for the line ℓ through the points $A = (-2, 1, 0)$ and $B = (4, 3, -2)$.

Solution. Let $A = (-2, 1, 0)$ and $B = (4, 3, -2)$ have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Then the line ℓ through A and B has direction

$$\mathbf{u} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -2 \end{pmatrix},$$

and so has parametric equations

$$\left.\begin{aligned} x &= -2 + 6\lambda \\ y &= 1 + 2\lambda \\ z &= - 2\lambda \end{aligned}\right\}.$$ ✓

(Here, it is valid to take any nonzero scalar multiple of $\mathbf{u}$ as 'the' direction vector. In particular, it is perfectly valid to use $-\mathbf{u} = \mathbf{a} - \mathbf{b}$ for this, which would negate coëfficients of λ seen above. Also, because of common factor 2 in the entries of $\mathbf{b} - \mathbf{a}$, we can use $\frac{1}{2}(\mathbf{b} - \mathbf{a})$, which would give an answer

$$\left.\begin{aligned} x &= -2 + 3\lambda \\ y &= 1 + \lambda \\ z &= - \lambda \end{aligned}\right\}.$$

Changing our 'favourite' point on the line [we are using A here] gives further different solutions.)

Lose 2 marks if the brace is omitted. No marks if you have not written down parametric equations, or if your answer is ambiguous.

Question 3 (10 marks). Let $A = (-1, 1, 2)$ and $B = (-2, 3, 4)$, and let ℓ be the unique line on which A and B lie. Let P be the point on ℓ that is *not* between A and B and such that the distance from P to B is half the distance from A to B. (Thus $|\overrightarrow{AP}| = \frac{3}{2}|\overrightarrow{AB}|$.) Determine the position vector $\mathbf{p}$ of P. [For notational convenience, A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively.]

Solution. The formula is $\mathbf{p} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$, where here $\lambda = \frac{3}{2}$. Thus we get that

$$\begin{aligned}\mathbf{p} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \frac{3}{2}\left[\begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}\right] &= \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -\frac{3}{2} \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{5}{2} \\ 4 \\ 5 \end{pmatrix}. \checkmark\end{aligned}$$

Lose 5 marks for each incorrect coördinate; lose 5 marks if I spot any nonsense such as you equating a point and a vector (to a minimum of 0 marks).

Question 4 (10 marks). Define carefully, but **without using coördinates**, the *scalar product* (or *dot product*) $\mathbf{u}\cdot\mathbf{v}$ of two vectors $\mathbf{u}$ and $\mathbf{v}$.

Solution. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ then we define $\mathbf{u}\cdot\mathbf{v} := 0$. Else $\mathbf{u}\cdot\mathbf{v} := |\mathbf{u}||\mathbf{v}|\cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. $\checkmark$

This question was answered extremely poorly, with most of you scoring zero (see the marking scheme below). In particular, the cases $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ were rarely mentioned, which is vital for a proper definition of $\mathbf{u}\cdot\mathbf{v}$. Also, "$\mathbf{u}\cdot\mathbf{v} := |\mathbf{u}||\mathbf{v}|\cos\theta$" introduces a hitherto undefined entity, namely θ, so you should specify what θ is, rather than hoping that the reader would guess. Lose 5 marks if the cases $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ are omitted, or if θ is not defined. (So lose 10 marks if both of these infractions occur.)

Question 5 (20 marks). Apply Gaußian elimination to transform the following system of linear equations into echelon form (indicate this clearly), followed by back substitution to determine their full solution set. [You may assume that these are linear equations over $\mathbb{R}$ in the variables x, y and z.]

$$\left.\begin{aligned} y + 2z &= -1 \\ x - 2y + 3z &= -3 \\ -3x + 6y + 9z &= -9 \end{aligned}\right\}.$$

Solution. First, we swap Equations 1 and 2 to get

$$\left.\begin{aligned} x - 2y + 3z &= -3 \\ y + 2z &= -1 \\ -3x + 6y + 9z &= -9 \end{aligned}\right\}.$$

Next, we add 3 times Equation 1 to Equation 3 to get

$$\left.\begin{aligned} x - 2y + 3z &= -3 \\ y + 2z &= -1 \\ 18z &= -18 \end{aligned}\right\},$$

which is now in echelon form. For the back substitution, we observe that all the variables are leading in one of the equations. From $18z = -18$, we get $z = -1$. Then $y + 2z = -1$ gives $y = -1 - 2(-1) = 1$, and then $x - 2y + 3z = -3$ gives $x = 2y - 3z - 3 = 2$. So their solution set is $\{(2, 1, -1)\}$.✓ (This is a point in $\mathbb{R}^3$.)

This question was answered poorly, though not quite so poorly as Question 4. The 20 marks were divided as 10 marks for the Gaußian elimination and 10 for the back substitution, with a minimum of 0 for each part (to prevent some of you getting large negative marks). Infractions included an incorrect step in the Gaußian elimination (lose 5 marks), and not specifying your solution correctly as a set (lose 3 marks).

Question 6 (20 marks). The four vertices of a regular tetrahedron of side length 2 can be taken to have position vectors $\mathbf{0}$, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, where $\mathbf{a} = 2\mathbf{i}$, $\mathbf{b} = \mathbf{i} + \sqrt{3}\mathbf{j}$ and $\mathbf{c}$ is dealt with below.

(a) **(5 marks)** Calculate the vector product $\mathbf{a} \times \mathbf{b}$.

(b) **(5 marks)** Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors of length 2 at an angle of $\frac{\pi}{3}$ to each other. Calculate $\mathbf{u}\cdot\mathbf{v}$. [Hint: $\mathbf{a}$ and $\mathbf{b}$ are two such vectors.]

(c) **(10 marks)** Given that $\mathbf{c}$ has length 2 and has angle $\frac{\pi}{3}$ with both $\mathbf{a}$ and $\mathbf{b}$, determine the two possibilities for $\mathbf{c}$.

Solution.

(a) $\mathbf{a} \times \mathbf{b} = (2\mathbf{i}) \times (\mathbf{i} + \sqrt{3}\mathbf{j}) = 2(\mathbf{i} \times \mathbf{i}) + 2\sqrt{3}(\mathbf{i} + \mathbf{j}) = \mathbf{0} + 2\sqrt{3}\mathbf{k} = 2\sqrt{3}\mathbf{k}.$ ✓

(b) We have $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \frac{\pi}{3} = 2.2.\frac{1}{2} = 2.$ ✓

(c) Let $\mathbf{c} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. From $\mathbf{a}\cdot\mathbf{c} = 2$ get $2a = 1$, so that $a = 1$. Now $\mathbf{b}\cdot\mathbf{c} = 2$ gives $1 + \sqrt{3}b = 2$, so that $b = \frac{1}{\sqrt{3}}$. Now $|\mathbf{c}| = 2$ gives

$$4 = |\mathbf{c}|^2 = a^2 + b^2 + c^2 = 1 + \tfrac{1}{3} + c^2,$$

and so $c^2 = \frac{8}{3}$, and thus $c = \pm\frac{2\sqrt{2}}{\sqrt{3}}$. Therefore $\mathbf{c} = \mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} \pm \frac{2\sqrt{2}}{\sqrt{3}}\mathbf{k}.$ ✓

I was disappointed that no-one managed to answer Part (c) correctly, which really is not that hard, though I was not expecting too many correct answers. I gave 5 marks if the first two coördinates of $\mathbf{c}$ were correct. The only answer I accepted for Part (b) was “2”.

Question 7 (10 marks). Exactly which of the following statements are true, and exactly which are false? [A statement of the form "s_1 and s_2 are true" will *not* be taken to mean that you think s_3, s_4 and s_5 are false.]

(a) If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are vectors such that $\mathbf{u}$ is orthogonal to $\mathbf{v}$ and $\mathbf{v}$ is orthogonal to $\mathbf{w}$, then $\mathbf{u}$ is orthogonal to $\mathbf{w}$.

(b) If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are coplanar then $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u}\times\mathbf{v}$ are coplanar.

(c) $\mathbf{v}\times\mathbf{u} = \mathbf{u}\times\mathbf{v}$ for all vectors $\mathbf{u}$ and $\mathbf{v}$.

(d) $\mathbf{v}\times\mathbf{u} \neq \mathbf{u}\times\mathbf{v}$ for all vectors $\mathbf{u}$ and $\mathbf{v}$.

(e) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{w}\cdot(\mathbf{v}\times\mathbf{u})$ for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

Solution. They are all false.✓ Possible counterexamples are as follows.

(a) $\mathbf{u}=\mathbf{i}$, $\mathbf{v}=\mathbf{j}$ and $\mathbf{w}=\mathbf{i}+\mathbf{k}$.

(b) $\mathbf{u}=\mathbf{i}$, $\mathbf{v}=\mathbf{j}$ and $\mathbf{w}=\mathbf{i}+\mathbf{j}$. (We have $\mathbf{u}\times\mathbf{v}=\mathbf{k}$.)

(c) $\mathbf{u}=\mathbf{i}$, $\mathbf{v}=\mathbf{j}$. We have $\mathbf{v}\times\mathbf{u} = -\mathbf{k} \neq \mathbf{k} = \mathbf{u}\times\mathbf{v}$.

(d) $\mathbf{u}=\mathbf{v}=\mathbf{0}$. We have $\mathbf{v}\times\mathbf{u}=\mathbf{u}\times\mathbf{v}=\mathbf{0}$.

(e) $\mathbf{u}=\mathbf{i}$, $\mathbf{v}=\mathbf{j}$ and $\mathbf{w}=\mathbf{k}$. We have $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 1 \neq -1 = \mathbf{w}\cdot(\mathbf{v}\times\mathbf{u})$.

The solutions you produced for this were little (if at all) better than using coin tosses with a fair coin to decide your answers. Marking scheme: 10 marks if all correct; 3 marks if 4 correct; otherwise (including cases of ambiguity) 0 marks.

Question 8 (10 marks). Let Π be the plane (in $\mathbb{R}^3$) with the equation $2x+y-2z=4$. Determine a point (called P) on Π and a nonzero vector (called $\mathbf{n}$) orthogonal to Π. Use this information to determine a point (called Q) at distance 6 from Π. [You may specify $\mathbf{p}$ and $\mathbf{q}$ instead of P and Q, where $\mathbf{p}$ and $\mathbf{q}$ are (respectively) the position vectors of P and Q.]

Solution. (NB: Answer not unique.) One reads off the coëfficients of x, y, z in the equation of Π to get:

$$\mathbf{n} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix},$$

though any nonzero scalar multiple of this is also correct. One method suggested in lectures to get a point on Π was to set two of x, y, z to 0, and see what the third one becomes. This would give $P=(2,0,0)$, $(0,4,0)$ or $(0,0,-2)$, though there are other possibilities, such as $(2,2,1)$.

Now $|\mathbf{n}|=3$, so we add $2\mathbf{n}$ (or $-2\mathbf{n}$) to the position vector of P to get the position vector of Q. This would give $Q=(6,2,-4)$ (or $(-2,-2,4)$) for the case $P=(2,0,0)$.✓ (Correct answers must have $\mathbf{q}\cdot\mathbf{n}=22$ or -14, with $\mathbf{n}$ as above.)

Basic marking scheme: 5 marks for both $\mathbf{n}$ <u>and</u> $\mathbf{p}$; 5 marks for $\mathbf{q}$.