MTH4103: Geometry I

Queen Mary
University of London

Mid-term Test 2011 Solutions

Duration: 40 minutes

Date and Time: 10th November 2011, in the hour 13:00–14:00

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The duration of this test is **40 minutes**. Answer **all** 10 questions. Each question is worth an equal number of marks (10 each, though you can only score 0, 5 or 10 marks for most questions). Apart from Question 5, only the final answer to a question will be marked, so indicate this answer clearly. Simplify your answers as much as possible.

Calculators are **not** permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Answer all questions in the spaces provided. You may do additional rough work on the backs of the question sheets or at the end, but this will not be looked at.

Do **not** turn over until an invigilator instructs you to do so.

For Examiner's use only

Question	Mark	Question	Mark
1	10	6	10
2	10	7	10
3	10	8	10
4	10	9	10
5	10	10	10
		Total (%)	100

Question 1. Let $\mathbf{a} = \begin{pmatrix} 2 \\ -9 \\ -6 \end{pmatrix}$. Determine $|\mathbf{a}|$.

Solution. We have $|\mathbf{a}| = \sqrt{2^2 + (-9)^2 + (-6)^2} = \sqrt{4 + 81 + 36} = \sqrt{121} = 11.$✓

Question 2. Determine Cartesian equations for the line through the points $(5, 0, 4)$ and $(1, -1, 5)$.

Solution. Let $A = (5, 0, 4)$ and $B = (1, -1, 5)$ have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Then the line ℓ through A and B has direction

$$\mathbf{u} = \mathbf{a} - \mathbf{b} = \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix},$$

and so has parametric equations

$$\left.\begin{aligned} x &= 5 + 4\lambda \\ y &= \lambda \\ z &= 4 - \lambda \end{aligned}\right\},$$

and thus Cartesian equations

$$\frac{x-5}{4} = y = \frac{z-4}{-1}.$$✓

Question 3. Let $A = (-2, -3, -1)$ and $B = (2, 5, 0)$, and let P be one quarter of the way along the line segment from A to B. (Thus $|\overrightarrow{AP}| = \frac{1}{4}|\overrightarrow{AB}|$.) Determine the position vector $\mathbf{p}$ of P. [For notational convenience, A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively.]

Solution. The formula is $\mathbf{p} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$, where here $\lambda = \frac{1}{4}$. Thus we get that

$$\begin{aligned}\mathbf{p} = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} + \frac{1}{4}\left[\begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}\right] &= \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ \frac{1}{4} \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -\frac{3}{4} \end{pmatrix}. \checkmark\end{aligned}$$

Question 4. Determine the cosine of the angle θ between the vectors $\mathbf{p} = \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$. Remember to simplify your answer as much as possible.

Solution. We have $|\mathbf{p}| = \sqrt{18} = 3\sqrt{2}$, $|\mathbf{q}| = \sqrt{6} = \sqrt{3}\sqrt{2}$ and $\mathbf{p}\cdot\mathbf{q} = 1 - 2 + 4 = 3$. Therefore, we have

$$\cos\theta = \frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{p}||\mathbf{q}|} = \frac{3}{3\sqrt{2}\sqrt{3}\sqrt{2}} = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}. \checkmark$$

Question 5. Apply Gaußian elimination to transform the following system of linear equations into echelon form. [You do not need to determine the solutions of the system.]

$$\left.\begin{aligned} x - 2y + 7z &= -7 \\ -3x + 6y + 7z &= -7 \\ y + 7z &= -7 \end{aligned}\right\}.$$

Solution. First, we add 3 times Equation 1 to Equation 2 to get

$$\left.\begin{aligned} x - 2y + 7z &= -7 \\ 28z &= -28 \\ y + 7z &= -7 \end{aligned}\right\}.$$

Next we swap Equations 2 and 3 to get

$$\left.\begin{aligned} x - 2y + 7z &= -7 \\ y + 7z &= -7 \\ 28z &= -28 \end{aligned}\right\},$$

which is now in echelon form.✓

Question 6. Let Π_1 be the plane with Cartesian equation

$$x - 2y - 5z = 23.$$

Choose a plane Π_2 having empty intersection with Π_1, and write down two distinct points on Π_2. [There are many possible answers here: any answer consistent with what I have asked you to do will be marked correct.]

Solution. Two points are $(2, 0, 0)$ and $(0, -1, 0)$.✓ (These are on the plane Π_2 with Cartesian equation $x - 2y - 5z = 2$.)

Question 7. Calculate the volume of the parallelepiped determined by the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ -4 \\ -1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$.

Solution. We have

$$\mathbf{b}\times\mathbf{c} = \begin{pmatrix} 0 \\ -4 \\ -1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} = \begin{vmatrix} -4 & 0 \\ -1 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & -1 \\ -1 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & -1 \\ -4 & 0 \end{vmatrix}\mathbf{k} = \begin{pmatrix} -12 \\ 1 \\ -4 \end{pmatrix}.$$

Therefore $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = -24 + 3 + 16 = -5$, and so the volume is $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| = |-5| = 5$. ✓

Question 8. Determine the distance from the point $Q = (2, 3, 1)$ to the plane Π having Cartesian equation $2x - y + 2z = 4$.

Solution. The formula is $|\mathbf{q}\cdot\mathbf{n} - d|/|\mathbf{n}|$ where $\mathbf{q}$ is the position vector of Q, $\mathbf{n}$ is a normal to Π, and d is the right-hand side of the Cartesian equation corresponding to $\mathbf{n}$. So here we have

$$\mathbf{q} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \quad \text{and} \quad d = 4.$$

So the distance is

$$\frac{|\mathbf{q}\cdot\mathbf{n} - d|}{|\mathbf{n}|} = \frac{|3-4|}{3} = \frac{1}{3}.$$ ✓

Question 9. Exactly which of the following statements are true? [Ambiguous answers will score 0 marks. Otherwise you get 10 marks for getting all five correct; 3 marks for four correct; and 0 marks for three or fewer correct.]

(a) If $\mathbf{v} \times \mathbf{u} = \mathbf{u} \times \mathbf{v}$ then $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

(b) If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is a right-handed triple then $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} \times \mathbf{v}$ are not coplanar.

(c) If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are coplanar then $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} \times \mathbf{v}$ are coplanar.

(d) $\mathbf{u} \times (2\mathbf{v} + \mathbf{u}) = -2(\mathbf{v} \times \mathbf{u})$ for all vectors $\mathbf{u}$ and $\mathbf{v}$.

(e) $\mathbf{u} + (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} + \mathbf{v}) \times (\mathbf{u} + \mathbf{w})$ for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

Solution. (a), (b) and (d) are true, (c) and (e) are false.✓

Question 10. Suppose $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors with $|\mathbf{u} \times \mathbf{v}| = -\sqrt{3}(\mathbf{u} \cdot \mathbf{v})$. Is this possible? If not, why not? If so, determine the angle (or possible angles) θ between $\mathbf{u}$ and $\mathbf{v}$. [You will gain some marks if you have determined both $\sin\theta$ and $\cos\theta$.]

Solution. If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, we let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. So we have $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$ (even when $\mathbf{u}$, $\mathbf{v}$ parallel). Also $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$. So we must solve $|\mathbf{u}||\mathbf{v}|\sin\theta = -\sqrt{3}|\mathbf{u}||\mathbf{v}|\cos\theta$. Since $\mathbf{u}$, $\mathbf{v}$ nonzero, we can cancel the $|\mathbf{u}||\mathbf{v}|$'s on both sides to get:

$$\sin\theta = -\sqrt{3}\cos\theta.$$

Squaring both sides gives $\sin^2\theta = 3\cos^2\theta$. So now $1 = \sin^2\theta + \cos^2\theta = 4\cos^2\theta$. Thus $\cos^2\theta = \frac{1}{4}$ and $\sin^2\theta = \frac{3}{4}$. But $0 \leqslant \theta \leqslant \pi$, and so $\sin\theta \geqslant 0$, and so $\sin\theta = \frac{\sqrt{3}}{2}$. Thus $\cos\theta = (\sin\theta)/(-\sqrt{3}) = -\frac{1}{2}$. The only value of θ satisfying the above is $\theta = \frac{2\pi}{3}$ (120 degrees).✓