MTH4103: Geometry I

Queen Mary
University of London

Mid-term Test 2011 Solutions

Duration: 40 minutes

Date and Time: 10th November 2011, in the hour 13:00–14:00

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The duration of this test is **40 minutes**. Answer **all** 10 questions. Each question is worth an equal number of marks (10 each, though you can only score 0, 5 or 10 marks for most questions). Apart from Question 5, only the final answer to a question will be marked, so indicate this answer clearly. Simplify your answers as much as possible.

Calculators are **not** permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Answer all questions in the spaces provided. You may do additional rough work on the backs of the question sheets or at the end, but this will not be looked at.

Do **not** turn over until an invigilator instructs you to do so.

Question 1. Let $\mathbf{a} = \begin{pmatrix} 2 \\ -9 \\ -6 \end{pmatrix}$. Determine $|\mathbf{a}|$.

Solution. We have $|\mathbf{a}| = \sqrt{2^2 + (-9)^2 + (-6)^2} = \sqrt{4 + 81 + 36} = \sqrt{121} = 11$.

Question 2. Determine Cartesian equations for the line through the points $(5, 0, 4)$ and $(1, -1, 5)$.

Solution. Let $A = (5, 0, 4)$ and $B = (1, -1, 5)$ have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Then the line ℓ through A and B has direction

$$\mathbf{u} = \mathbf{a} - \mathbf{b} = \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix},$$

and so has parametric equations

$$\left.\begin{aligned} x &= 5 + 4\lambda \\ y &= \lambda \\ z &= 4 - \lambda \end{aligned}\right\},$$

and thus Cartesian equations

$$\frac{x-5}{4} = y = \frac{z-4}{-1}.$$

Other solutions are possible by using different direction vectors, such as $\mathbf{v} = \mathbf{b} - \mathbf{a} = -\mathbf{u}$, and/or by using a different fixed point on the line, such as B. Using B and the direction $-\mathbf{v}$ would give Cartesian equations $\frac{x-1}{-4} = \frac{y+1}{-1} = z - 5$. You should convince yourselves that the new equations are equivalent to the ones I gave above. The most common correct answers were something like one of

$$\frac{x-5}{-4} = \frac{y}{-1} = \frac{z-4}{1} \quad \text{or} \quad \frac{x-5}{-4} = -y = z - 4 \quad \text{or} \quad \frac{x-5}{4} = y = 4 - z.$$

Question 3. Let $A = (-2, -3, -1)$ and $B = (2, 5, 0)$, and let P be one quarter of the way along the line segment from A to B. (Thus $|\overrightarrow{AP}| = \frac{1}{4}|\overrightarrow{AB}|$.) Determine the position vector $\mathbf{p}$ of P. [For notational convenience, A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively.]

Solution. The formula is $\mathbf{p} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$ or $\mathbf{p} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$, where here $\lambda = \frac{1}{4}$. So here, using the first of these formulae, we get that

$$\begin{aligned}\mathbf{p} = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} + \frac{1}{4}\left[\begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}\right] &= \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ \frac{1}{4} \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -\frac{3}{4} \end{pmatrix}.\end{aligned}$$

Question 4. Determine the cosine of the angle θ between the vectors $\mathbf{p} = \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$. Remember to simplify your answer as much as possible.

Solution. We have $|\mathbf{p}| = \sqrt{18} = 3\sqrt{2}$, $|\mathbf{q}| = \sqrt{6} = \sqrt{3}\sqrt{2}$ and $\mathbf{p}\cdot\mathbf{q} = 1 - 2 + 4 = 3$. Therefore, we have

$$\cos\theta = \frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{p}||\mathbf{q}|} = \frac{3}{3\sqrt{2}\sqrt{3}\sqrt{2}} = \frac{1}{2\sqrt{3}}.$$

Also acceptable is $\frac{\sqrt{3}}{6}$ and, more marginally, $\frac{1}{\sqrt{12}}$ and $\frac{\sqrt{12}}{12}$. (Actually, although I accepted $\frac{\sqrt{12}}{12}$ I cannot say I like it much since a factor of 2 can clearly cancelled from the numerator [top] and denominator [bottom].)

Question 5. Apply Gaußian elimination to transform the following system of linear equations into echelon form. [You do not need to determine the solutions of the system.]

$$\left.\begin{aligned} x - 2y + 7z &= -7 \\ -3x + 6y + 7z &= -7 \\ y + 7z &= -7 \end{aligned}\right\}.$$

Solution. First, we add 3 times Equation 1 to Equation 2 to get

$$\left.\begin{aligned} x - 2y + 7z &= -7 \\ 28z &= -28 \\ y + 7z &= -7 \end{aligned}\right\}.$$

Next we swap Equations 2 and 3 to get

$$\left.\begin{aligned} x - 2y + 7z &= -7 \\ y + 7z &= -7 \\ 28z &= -28 \end{aligned}\right\},$$

which is now in echelon form. (The unique solution of this system of equations is $x = 0$, $y = 0$, $z = -1$, but the question did not require you to calculate this.)

Note that the Geometry I version of the Gaußian elimination algorithm does not permit variation, so your working had to be exactly as above (including the braces) in order to gain full marks. Also, marks were lost for missing a step out, as that made the method you were using unclear.

Question 6. Let Π_1 be the plane with Cartesian equation

$$x - 2y - 5z = 23.$$

Choose a plane Π_2 having empty intersection with Π_1, and write down two distinct points on Π_2. [There are many possible answers here: any answer consistent with what I have asked you to do will be marked correct.]

Solution. The only way we can have $\Pi_1 \cap \Pi_2 = \varnothing$ is if Π_2 is parallel to but not coincident with Π_1. So Π_2 has Cartesian equation $x - 2y - 5z = c$, for some constant $c \neq 23$. (Any nonzero multiple of the equation $x - 2y - 5z = c$ is also valid as an equation for Π_2.) Now $\mathbf{n} = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}$ is a vector orthogonal to both Π_1 and Π_2, and we simply pick points A and B whose position vectors $\mathbf{a}$ and $\mathbf{b}$ satisfy $\mathbf{a} \neq \mathbf{b}$ and $\mathbf{a}\cdot\mathbf{n} = \mathbf{b}\cdot\mathbf{n} = c$. Concretely, we can pick $c = 0$, and let $A = (0, 0, 0)$ and $B = (2, 1, 0)$. (Why make life hard for yourself?) Alternatively, we can pick $c = 2$, and let $A = (2, 0, 0)$ and $B = (0, -1, 0)$.

Question 7. Calculate the volume of the parallelepiped determined by the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ -4 \\ -1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$.

Solution. We have

$$\mathbf{b}\times\mathbf{c} = \begin{pmatrix} 0 \\ -4 \\ -1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} = \begin{vmatrix} -4 & 0 \\ -1 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & -1 \\ -1 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & -1 \\ -4 & 0 \end{vmatrix}\mathbf{k} = \begin{pmatrix} -12 \\ 1 \\ -4 \end{pmatrix}.$$

Therefore $\mathbf{a}\cdot(\mathbf{b} \times \mathbf{c}) = -24 + 3 + 16 = -5$, and so the volume is $|\mathbf{a}\cdot(\mathbf{b} \times \mathbf{c})| = |-5| = 5$. (Also, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is a left-handed triple, since $\mathbf{a}\cdot(\mathbf{b} \times \mathbf{c}) < 0$.)

Nonsensical answers, such as the volume being negative, attracted more punishment than mere arithmetical slips.

Question 8. Determine the distance from the point $Q = (2, 3, 1)$ to the plane Π having Cartesian equation $2x - y + 2z = 4$.

Solution. The formula is $|\mathbf{q}\cdot\mathbf{n} - d|/|\mathbf{n}|$ where $\mathbf{q}$ is the position vector of Q, $\mathbf{n}$ is a normal to Π, and d is the right-hand side of the Cartesian equation corresponding to $\mathbf{n}$. So here we have

$$\mathbf{q} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \quad \text{and} \quad d = 4.$$

So the distance is

$$\frac{|\mathbf{q}\cdot\mathbf{n} - d|}{|\mathbf{n}|} = \frac{|3-4|}{3} = \frac{1}{3}.$$

Nonsensical answers, such as the distance being negative, attracted more punishment than mere arithmetical slips.

Question 9. Exactly which of the following statements are true? [Ambiguous answers will score 0 marks. Otherwise you get 10 marks for getting all five correct; 3 marks for four correct; and 0 marks for three or fewer correct.]

(a) If $\mathbf{v} \times \mathbf{u} = \mathbf{u} \times \mathbf{v}$ then $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

(b) If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is a right-handed triple then $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} \times \mathbf{v}$ are not coplanar.

(c) If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are coplanar then $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} \times \mathbf{v}$ are coplanar.

(d) $\mathbf{u} \times (2\mathbf{v} + \mathbf{u}) = -2(\mathbf{v} \times \mathbf{u})$ for all vectors $\mathbf{u}$ and $\mathbf{v}$.

(e) $\mathbf{u} + (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} + \mathbf{v}) \times (\mathbf{u} + \mathbf{w})$ for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

Solution. (a), (b) and (d) are true, (c) and (e) are false. Note that simply saying that (a), (b) and (d) are true is ambiguous, as it says nothing about (c) and (e).

For (a) see Coursework 5, Feedback Question Part (d).

For (b), if $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is right-handed then $\mathbf{u}$, $\mathbf{v}$ are not collinear, and so $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} \times \mathbf{v}$ is right-handed by definition of the cross product, so they are certainly not coplanar.

For (c), we take $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{j}$, $\mathbf{w} = \mathbf{i}+\mathbf{j}$, so that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are coplanar, while $\mathbf{u}\times\mathbf{v} = \mathbf{k}$, so that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} \times \mathbf{v}$ are not coplanar. (In fact, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} \times \mathbf{v}$ are coplanar if and only if $\mathbf{u}$, $\mathbf{v}$ are collinear.)

For (d), we use theorems in our notes to establish that $\mathbf{u}\times(2\mathbf{v}+\mathbf{u}) = 2(\mathbf{u}\times\mathbf{v})+\mathbf{u}\times\mathbf{u} = 2(\mathbf{u} \times \mathbf{v}) + \mathbf{0} = -2(\mathbf{v} \times \mathbf{u})$.

For (e), we see that the equality does not hold for say $\mathbf{u} = \mathbf{i}$ and $\mathbf{v} = \mathbf{w} = \mathbf{0}$, with the left-hand side being $\mathbf{i}$ and the right-hand side being $\mathbf{0}$ in this case.

Question 10. Suppose $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors with $|\mathbf{u}\times\mathbf{v}| = -\sqrt{3}(\mathbf{u}\cdot\mathbf{v})$. Is this possible? If not, why not? If so, determine the angle (or possible angles) θ between $\mathbf{u}$ and $\mathbf{v}$. [You will gain some marks if you have determined both $\sin\theta$ and $\cos\theta$.]

Solution. If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, we let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. So we have $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$ (even when $\mathbf{u}$, $\mathbf{v}$ parallel). Also $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$. So we must solve $|\mathbf{u}||\mathbf{v}| \sin\theta = -\sqrt{3}|\mathbf{u}||\mathbf{v}| \cos\theta$. Since $\mathbf{u}$, $\mathbf{v}$ nonzero, we can cancel the $|\mathbf{u}||\mathbf{v}|$'s on both sides to get:

$$\sin\theta = -\sqrt{3}\cos\theta.$$

Squaring both sides gives $\sin^2\theta = 3\cos^2\theta$. So now $1 = \sin^2\theta + \cos^2\theta = 4\cos^2\theta$. Thus $\cos^2\theta = \frac{1}{4}$ and $\sin^2\theta = \frac{3}{4}$. But $0 \leqslant \theta \leqslant \pi$, and so $\sin\theta \geqslant 0$, and so $\sin\theta = \frac{\sqrt{3}}{2}$. Thus $\cos\theta = (\sin\theta)/(-\sqrt{3}) = -\frac{1}{2}$. The only value of θ satisfying the above is $\theta = \frac{2\pi}{3}$ (120 degrees).

Aliter: After establishing that $\sin\theta = -\sqrt{3}\cos\theta$, we know that $\tan\theta = \frac{\sin\theta}{\cos\theta} = -\sqrt{3}$, which only happens at $\theta = \frac{2\pi}{3}$ for θ in the range $0 \leqslant \theta \leqslant \pi$, as it must be. (Note, however, that $\arctan(-\sqrt{3}) = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$.) But we have divided by $\cos\theta$, and must consider if anything untoward happens when $\cos\theta = 0$, that is when $\theta = \frac{\pi}{2}$. However, $\sin\frac{\pi}{2} = 1$ and $\cos\frac{\pi}{2} = 0$, so $\theta \neq \frac{\pi}{2}$ here.