Geometry I, 2007 : Mid-term test solutions

The duration of this test is **40 minutes**. Answer **all** 10 questions. Each question is worth 1 mark. Only the final answer to a question will be marked, so indicate this answer clearly. Calculators are **not** allowed.

Answer all questions in the spaces provided. You may do additional rough work on the backs of the question sheets, but this will not be looked at.

1. Let $A = (-3, 1, 1)$, $B = (2, -1, 3)$. Determine the vector represented by $\overrightarrow{AB}$.

Where $\underline{a}$ is the position vector of A and $\underline{b}$ is the position vector of B, the vector represented by $\overrightarrow{AB}$ is $\underline{b} - \underline{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 2 \end{pmatrix}$.

2. Let $\underline{u} = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$. Determine the vector of length 1 having direction opposite that of $\underline{u}$.

This vector is

$$-\frac{1}{|\underline{u}|}\underline{u} = -\frac{1}{\sqrt{11}} \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/\sqrt{11} \\ -1/\sqrt{11} \\ -1/\sqrt{11} \end{pmatrix}.$$

3. Determine Cartesian equations for the line through the point $(-3, 1, 1)$ and in the direction of the vector $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$.

A vector equation for this line is $\underline{r} = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, giving parametric equations: $\begin{array}{rcrcr} x & = & -3 & + & 2\lambda \\ y & = & 1 & - & \lambda \\ z & = & 1 & + & 3\lambda \end{array}$.

Thus Cartesian equations for the line are:

$$\frac{x+3}{2} = \frac{y-1}{-1} = \frac{z-1}{3}.$$

4. Determine the cosine of the angle between the vectors $\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$.

Let $\underline{u} = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, and suppose these vectors are at angle θ. Then

$$\cos\theta = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}||\underline{v}|} = \frac{-4}{\sqrt{11}\sqrt{14}} = \frac{-4}{\sqrt{154}}.$$

5. Determine a Cartesian equation for the plane through the point $(-3, 1, 1)$ and orthogonal to the vector $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$.

This equation is calculated as $2x - y + 3z = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, which gives

$$2x - y + 3z = -4.$$

6. Determine all solutions of the following system of linear equations in x, y, z:

$$\begin{cases} x - y - z = -2 \\ -3x + 3y + 4z = 7 \\ 2x - 3y + z = -2 \end{cases}.$$

Adding 3 times the first equation to the second and -2 times the first to the third, we get $\begin{cases} x - y - z = -2 \\ z = 1 \\ -y + 3z = 2 \end{cases}$, and interchanging the second and third equations gives $\begin{cases} x - y - z = -2 \\ -y + 3z = 2 \\ z = 1 \end{cases}$, which is in echelon form.

Applying back substitution, $z = 1$, $-y + 3 = 2$ and so $y = 1$, $x - 1 - 1 = -2$ and so $x = 0$.

Thus $x = 0, y = 1, z = 1$ is the only solution.

7. Suppose $\underline{u}$ and $\underline{v}$ are non-zero vectors with $|\underline{u} \times \underline{v}| = -\underline{u} \cdot \underline{v}$. Is this possible? If not, why not? If so, determine the angle θ between $\underline{u}$ and $\underline{v}$.

We have $|\underline{u} \times \underline{v}| = -\underline{u} \cdot \underline{v}$ if and only if $|\underline{u}||\underline{v}| \sin\theta = -|\underline{u}||\underline{v}| \cos\theta$ for some θ with $0 \leq \theta \leq \pi$, and since $\underline{u}$ and $\underline{v}$ are non-zero vectors, this is equivalent to $\sin\theta = -\cos\theta$ for some θ with $0 \leq \theta \leq \pi$. This is certainly possible, and happens exactly when

$$\theta = \frac{3\pi}{4}.$$

8. Suppose $\underline{u}, \underline{v}, \underline{w}$ is a right-handed triple of vectors. Exactly which of the following are right-handed triples?

(a) $\underline{v} \times \underline{w}, \underline{w}, \underline{v}$

(b) $\underline{w}, \underline{v}, \underline{u}$

(c) $-\underline{v}, \underline{u}, -\underline{w}$

(d) $\underline{u}, -\underline{w}, \underline{v}$

(e) $\underline{v}, \underline{v} \times (-\underline{w}), \underline{w}$

The only right-handed triples are (d) and (e).

9. Determine the volume of a parallepiped with sides corresponding to $\underline{u} = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$, $\underline{v} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, and $\underline{w} = \begin{pmatrix} -2 \\ -1 \\ 4 \end{pmatrix}$.

We have $\underline{v} \times \underline{w} = \begin{vmatrix} -1 & -1 \\ 3 & 4 \end{vmatrix} \underline{i} - \begin{vmatrix} 2 & -2 \\ 3 & 4 \end{vmatrix} \underline{j} + \begin{vmatrix} 2 & -2 \\ -1 & -1 \end{vmatrix} \underline{k} = -\underline{i} - 14\underline{j} - 4\underline{k} = \begin{pmatrix} -1 \\ -14 \\ -4 \end{pmatrix}$, and the volume is $|\underline{u} \cdot (\underline{v} \times \underline{w})| = |(-3)(-1) - 14 - 4| = |-15| = 15$.

10. Consider the planes defined by $x - 2y + 2z = 3$ and by $y + z = 0$. Determine a non-zero vector parallel to both of these planes.

One method of solution proceeds by finding two distinct points A, B in the intersection of the two planes (this intersection is easy to compute and is a line), and then a vector parallel to both planes is represented by $\overrightarrow{AB}$.

Two such points are $A = (3, 0, 0)$ and $B = (-1, -1, 1)$. Then $\overrightarrow{AB}$ represents $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \\ 1 \end{pmatrix}$, which is parallel to both planes.

Alternatively, a vector parallel to both planes could be computed as $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \\ 1 \end{pmatrix}$. (Can you see why this works?)