Geometry I, 2006 : Mid-term test sample solutions

The duration of this test is **40 minutes**. Answer **all** 10 questions. Each question is worth 1 mark. Only the final answer to a question will be marked, so indicate this answer clearly. Calculators are **not** allowed.

1. Let $A = (1, 2, 3)$, $B = (2, -1, 4)$. Determine the vector represented by $\overrightarrow{AB}$.

This vector is $\underline{b} - \underline{a} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$.

(Throughout these solutions, the position vectors of points $A, B, C, \ldots$ are denoted by $\underline{a}, \underline{b}, \underline{c}, \ldots$.)

2. Let $A = (1, 2, 3)$, $B = (2, -1, 4)$. Determine the position vector of the point P on the line segment AB, such that $|\overrightarrow{AP}| = \frac{1}{2}|\overrightarrow{AB}|$.

$\underline{p} = (1 - \frac{1}{2})\underline{a} + \frac{1}{2}\underline{b} = \frac{1}{2}(\underline{a} + \underline{b}) = \frac{1}{2}\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 1/2 \\ 7/2 \end{pmatrix}$.

3. Let $A = (1, 2, 3)$, $B = (2, -1, 4)$. Determine parametric equations for the line through A and B.

This is the line through A in the direction of $\underline{b} - \underline{a}$, and so a vector equation for this line is
$\underline{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$. This gives the parametric equations:

$$\begin{array}{ccccc} x & = & 1 & + & \lambda \\ y & = & 2 & - & 3\lambda \\ z & = & 3 & + & \lambda \end{array}$$

4. Let $A = (1, 2, 3)$, $B = (2, -1, 4)$, $D = (2, 0, -3)$. Determine the point C such that $ABCD$ is a parallelogram.

For $ABCD$ to be a parallelogram, $\overrightarrow{DC}$ must represent the same vector as $\overrightarrow{AB}$. In other words, $\underline{c} - \underline{d} = \underline{b} - \underline{a}$, and so $\underline{c} = \underline{d} + \underline{b} - \underline{a}$.
Thus, $\underline{c} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -2 \end{pmatrix}$.
Therefore $C = (3, -3, -2)$.

5. Determine the cosine of the angle between the vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$.

Let $\underline{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$, and suppose these vectors are at angle θ. Then

$$\cos\theta = \frac{\underline{u}\cdot\underline{v}}{|\underline{u}||\underline{v}|} = \frac{12}{\sqrt{14}\sqrt{21}} = \frac{12}{7\sqrt{6}} = \frac{2\sqrt{6}}{7}.$$

6. Determine all solutions of the following system of linear equations in x, y, z:

$$\begin{cases} x & - & y & - & 2z & = & -1 \\ -3x & + & y & + & z & = & 2 \\ 2x & - & 2y & + & 4z & = & -2 \end{cases}.$$

This system is equivalent to

$$\begin{cases} x & - & y & - & 2z & = & -1 \\ & - & 2y & - & 5z & = & -1 \\ & & & & 8z & = & 0 \end{cases}.$$

Therefore $z = 0$, $-2y = -1$ and so $y = 1/2$, $x - 1/2 = -1$ and so $x = -1/2$.

Thus $x = -1/2, y = 1/2, z = 0$ is the only solution.

7. Determine the intersection of the plane defined by $x - 2y + 3z = 4$ with the line ℓ through the point $(1, 2, 3)$ and in the direction of $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$.

The line ℓ has vector equation $\underline{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$.
To find the intersection of the plane with ℓ, we solve $(1 + 3\lambda) - 2(2) + 3(3 + \lambda) = 4$. This gives $6\lambda = -2$, $\lambda = -1/3$. Substituting $\lambda = -1/3$ into the equation for ℓ, we deduce that the intersection of the plane and ℓ consists of the single point $(0, 2, 8/3)$.

8. Determine the vector product $\underline{u} \times \underline{v}$, where $\underline{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$.

$$\underline{u} \times \underline{v} = \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} \underline{i} - \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \underline{j} + \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \underline{k}$$
$$= 11\underline{i} + 2\underline{j} - 5\underline{k} = \begin{pmatrix} 11 \\ 2 \\ -5 \end{pmatrix}.$$

9. Exactly which of the following statements are true?

(a) If $\underline{u}$ and $\underline{v}$ are vectors such that $\underline{u} \times \underline{v} = \underline{0}$, then we must have $\underline{u} = \underline{0}$ or $\underline{v} = \underline{0}$.

(b) If $\underline{u}$ is a vector such that $\underline{u} \times \underline{v} = \underline{0}$ for every vector $\underline{v}$, then we must have $\underline{u} = \underline{0}$.

(c) $\underline{u} \times \underline{v} = \underline{v} \times \underline{u}$ for all vectors $\underline{u}, \underline{v}$.

(d) $(\underline{u} \times \underline{v}) \times \underline{w} = \underline{u} \times (\underline{v} \times \underline{w})$ for all vectors $\underline{u}, \underline{v}, \underline{w}$.

(e) $\underline{u} \cdot (\underline{v} \times \underline{w}) = \underline{w} \cdot (\underline{u} \times \underline{v})$ for all vectors $\underline{u}, \underline{v}, \underline{w}$.

The only true statements are (b) and (e).

10. Let $A = (1, 2, 3)$ and $B = (2, -1, 4)$. Determine a Cartesian equation for a plane through A and B and parallel to the vector $\underline{u} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$.

This is the plane through the point A and parallel to both $\underline{u}$ and the vector $\underline{v}$ represented by $\overrightarrow{AB}$.

Now $\underline{v} = \underline{b} - \underline{a} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$, and a vector equation for the plane is $\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}$, where $\underline{n} = \underline{u} \times \underline{v} = -9\underline{i} - 5\underline{j} - 6\underline{k}$. Thus a Cartesian equation for the plane is $-9x - 5y - 6z = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -9 \\ -5 \\ -6 \end{pmatrix} = -37$, equivalently, $9x + 5y + 6z = 37$.