Chapter 8

Determinants

The founder of the theory of determinants is usually taken to be Gottfried Wilhelm Leibniz (1646–1716), who also shares the credit for inventing calculus with Sir Isaac Newton (1643–1727)[1]. But the idea of 2×2 determinants goes back at least to the Chinese around 200 BC. The word *determinant* itself was first used in its present sense in 1812 by Augustin-Louis Cauchy (1789–1857), and he developed much of the general theory we know today.

8.1 Inverses of 2×2 matrices, and determinants

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix. Recall that the determinant of A is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = ad - cb.$$

We also denote this determinant by $\det(A)$ or $\det A$ or $|A|$. (Just as with other operators, such as cos, we avoid using the brackets except to resolve ambiguity. Thus we prefer $\cos\theta$ and $\det A$ in preference to $\cos(\theta)$ and $\det(A)$, and $\det A^\mathsf{T}$ means $\det(A^\mathsf{T})$.)

Examples. Let $A = \begin{pmatrix} 2 & 3 \\ 2 & -1 \end{pmatrix}$. Then $\det A = -2 - 6 = -8$. Also, we have

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2, \quad \det \mathrm{I}_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \quad \text{and} \quad \begin{vmatrix} 2 & -1 \\ -4 & 2 \end{vmatrix} = 4 - 4 = 0.$$

Lemma 8.1. Let A be any 2×2 matrix. Then $\det A^\mathsf{T} = \det A$.

Proof. If $A^\mathsf{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^\mathsf{T} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, and so $\det A^\mathsf{T} = ad - cb = \det A$. □

[1] His dates by the Gregorian calendar are 4th January 1643 – 31st March 1727. But the Julian calendar was in use in England at the time, and his Julian dates are 25th December 1642 – 20th March 1727. At the time, the year in England started on 25th March, so that legally he died in 1726.

Now let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and let $\Delta := \det A = ad - bc$. We note that if $\Delta \neq 0$ and

$$B = \begin{pmatrix} d/\Delta & -b/\Delta \\ -c/\Delta & a/\Delta \end{pmatrix}$$

then

$$AB = \begin{pmatrix} (ad-bc)/\Delta & (-ab+ba)/\Delta \\ (cd-dc)/\Delta & (-cb+da)/\Delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$BA = \begin{pmatrix} (da-bc)/\Delta & (db-bd)/\Delta \\ (-ca+ac)/\Delta & (-cb+ad)/\Delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so that A is invertible and $A^{-1} = B$. Thus if A is a 2×2 matrix and $\det A \neq 0$, then A is invertible. What if $\det A = 0$? We shall show that in this case A is *not* invertible, but first we show the following.

Theorem 8.2. If A and B are 2×2 matrices then $\det(AB) = \det(A)\det(B)$.

Notation. Our conventions for bracketing the determinant operator are similar to those used for trigonometric functions. Thus $\det AB$ means $\det(AB)$, even though $(\det A)B$ makes sense, and $\det A \det B$ means $\det(A)\det(B) = (\det A)(\det B)$, though $\det(A \det B)$ makes less sense since we write the scalar first when notating scalar multiplication of matrices. Similarly, if λ is a scalar then $\det \lambda A$ means $\det(\lambda A)$. Note that while notations like $\det 3A$ and $\det A \det B$ are often used, we should *never* write things like $\det -A$ or $\det -2A$ or $\det A + B$. Also, $\det AB$ is seldom seen.

Proof. We prove this by direct computation. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Then $AB = \begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}$ and

$$\begin{aligned} \det(AB) &= (ap+br)(cq+ds) - (cp+dr)(aq+bs) \\ &= apcq + apds + brcq + brds - cpaq - cpbs - draq - drbs \\ &= apds + brcq - cpbs - draq \\ &= (ad-cb)(ps-rq) \\ &= \det A \det B. \end{aligned}$$

□

Example. Let $A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$. Then

$$\det(AB) = \begin{vmatrix} -6 & 5 \\ 14 & 0 \end{vmatrix} = 0 - 70 = -70,$$

which is the same as $\det A \det B = (3-(-2))(-2-12) = 5 \times (-14) = -70$.

Theorem 8.3. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2-matrix. Then A is invertible if and only if $\det A \neq 0$, and when $\Delta := \det A \neq 0$, then $A^{-1} = \begin{pmatrix} d/\Delta & -b/\Delta \\ -c/\Delta & a/\Delta \end{pmatrix}$.

Proof. We have already seen that when $\Delta = \det A \neq 0$ then A is invertible with inverse as above. It therefore just remains to show that when $\Delta = 0$ then A is not invertible. So suppose that $\det A = 0$. Then $\det(AB) = \det A \det B = 0$ for every 2×2 matrix B. But since $\det \mathrm{I}_2 = 1$ this means there cannot exist a matrix B with $AB = \mathrm{I}_2$. Hence A is not invertible. $\square$

Examples. 1. Let $A = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$. Then $\det A = 0$, so A is not invertible.

2. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Then $\det A = -2 \neq 0$, so A is invertible, and

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}.$$

Let us check this:

$$AA^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{I}_2,$$

$$A^{-1}A = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{I}_2.$$

We conclude this section by introducing the *adjugate* of a 2×2 matrix. This might not seem to be a terribly exciting or useful concept for 2×2 matrices, but is far more useful for 3×3 matrices, and in general for $n \times n$ matrices ($n \in \mathbb{N}^+$). The adjugate of a square matrix A is denoted $\operatorname{adj} A$ (or $\operatorname{adj}(A)$), and for 2×2 matrices is defined by

$$\operatorname{adj} \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Direct calculation shows that we have $A(\operatorname{adj} A) = (\operatorname{adj} A)A = (\det A)\mathrm{I}_2$ for all 2×2 matrices A. Thus if $\det A \neq 0$ (precisely the conditions needed for A to be invertible) we have $A^{-1} = \frac{1}{\det A}(\operatorname{adj} A)$. These relations generalise to $n \times n$ matrices for all $n \geqslant 1$.

We emphasise that the adjugate of a matrix is not the same as its adjoint, and you must be careful not to confuse the two terms. (Unfortunately, what we now call the adjugate has in the past been termed the adjoint!) The *adjoint* of a matrix over $\mathbb{C}$ is defined to the transpose of its complex conjugate, and is denoted A^* or $A^\dagger$, though sometimes A^* simply means the complex conjugate of A. Thus the adjoint of matrix over $\mathbb{R}$ is the same as its transpose. (If you have not yet met complex numbers and the complex conjugate, you will have done so by the time you take your first year exams.)

8.2 Determinants of 3×3 matrices

Consider the 3×3 matrix

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

We define the *determinant* of A by:

$$\det A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} := \mathbf{a}\cdot(\mathbf{b} \times \mathbf{c})$$

where

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Thus

$$\begin{aligned} \det A &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \left(\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \times \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \right) \\ &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \left(\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \mathbf{k} \right) \\ &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\ &= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1). \end{aligned}$$

Example. Let

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 0 & -3 \\ -2 & 1 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} \det A &= 3 \begin{vmatrix} 0 & -3 \\ 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & -1 \\ 0 & -3 \end{vmatrix} \\ &= 3(3) - 2(3) - 2(-6) = 15. \end{aligned}$$

Remark. Notice that the right-hand side of our equation

$$\mathbf{a}\cdot(\mathbf{b} \times \mathbf{c}) = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

can be rearranged to give $a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3)$. Thus we can also write:

$$\begin{aligned} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ &= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \\ &= a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3). \end{aligned}$$

This formula ("expanding along the first row") is often used to define a 3×3 determinant (even by me!), and is more frequently seen than the triple scalar product definition ("expanding along the first column") used in this course. The reason for our seemingly strange approach is that we can easily prove many properties of 3×3 determinants using results we have already established for triple scalar products (see Theorem 8.5). Note that it also follows from this rearrangement that

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

In other words, the determinant of a 3×3 matrix A is equal to the determinant of its transpose A^{T}, a result we record as Theorem 8.4.

Another helpful rearrangement of the terms in $\det A$ is to group those terms with positive and negative coefficients together, to get

$$\det A = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1.$$

The products having positive coefficient correspond to diagonals going in the direction top-left to bottom-right in the diagram below.

$$\begin{array}{cc|ccc|cc} & & a_1 & a_2 & a_3 & & \\ & b_3 & b_1 & b_2 & b_3 & b_1 & \\ c_2 & c_3 & c_1 & c_2 & c_3 & c_1 & c_2 \end{array}$$

The top-right to bottom-left 'diagonals' in the above diagram correspond to those products with negative coefficient.

Theorem 8.4. For all 3×3 matrices A we have $\det A^{\mathsf{T}} = \det A$.

Proof. See above. □

Theorem 8.5. Let $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$. Then

(i) Interchanging two columns of A negates $\det A$. For example

$$\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\det A.$$

(ii) Multiplying any column of A by λ multiplies $\det A$ by λ.

(iii) Adding a scalar multiple of any column of A to any other column does not change the determinant.

Proof. These all follow very easily from properties we have already proved for the scalar triple product.

(i): This follows at once from Theorem 6.6, which gives $\det A = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = -\mathbf{a}\cdot(\mathbf{c}\times\mathbf{b}) = -\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c}) = -\mathbf{c}\cdot(\mathbf{b}\times\mathbf{a})$.

(ii): We have $\lambda \det A = \lambda(\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})) = (\lambda\mathbf{a})\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\cdot((\lambda\mathbf{b})\times\mathbf{c}) = \mathbf{a}\cdot(\mathbf{b}\times(\lambda\mathbf{c}))$, by Theorem 6.3 and Section 3.2.

(iii): Let B be the matrix obtained by adding λ times the second column to the first column of A. Then standard results on the dot and cross product, such as $\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c}) = 0$, give us $\det B = (\mathbf{a}+\lambda\mathbf{b})\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) + \lambda\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \det A$. The other cases are similar. □

Example. We can use these properties to compute determinants quickly.

$$\begin{vmatrix} 6 & 10 & 13 \\ 2 & 20 & 4 \\ -1 & -20 & -2 \end{vmatrix} = 10\begin{vmatrix} 6 & 1 & 13 \\ 2 & 2 & 4 \\ -1 & -2 & -2 \end{vmatrix} = 10\begin{vmatrix} 6 & 1 & 1 \\ 2 & 2 & 0 \\ -1 & -2 & 0 \end{vmatrix} = 10\begin{vmatrix} 0 & 1 & 1 \\ 2 & 2 & 0 \\ -1 & -2 & 0 \end{vmatrix}$$

$$= 10\left(0\begin{vmatrix} 2 & 0 \\ -2 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}\right) = 10(0(0) - 2(2) - 1(-2)) = -20.$$

Here to get the first equality we have applied Part (ii) of the theorem to take out a factor of 10 from the second column, to get the second equality we have applied Part (iii) of the theorem to subtract twice the first column from the third column, and for the 3rd equality we have applied Part (iii) to subtract 6 times the third column from the first column. (Note that we can use the theorem to simplify the matrix even more, so that we end up computing $\det \mathrm{I}_3$, which is 1.)

Remark. Since $\det A^{\mathsf{T}} = \det A$ for all 3×3 matrices A, Theorem 8.5 holds with the word 'column' replaced by 'row' throughout. Thus we can just as well use row operations as column operations when simplifying determinants.

The rest of this section is presented without proof. In particular, the next theorem is rather difficult to prove. (The one after that is not too hard.)

Theorem 8.6. For all 3×3 matrices A and B we have $\det(AB) = \det A \det B$.

Proof. In theory, we can expand $\det(AB)$, which has $3! \times 3^3 = 6 \times 27 = 162$ terms, perform all the cancellations to be left with $(3!)^2 = 6^2 = 36$ terms, and observe that this is $\det A \det B$. This is not very satisfactory, and new ideas are needed, which will also extend to general $n \times n$ determinants. However, this is beyond the scope of this course, though I may write up the proof as an appendix in the hope that some of you might understand it after having completed MTH4104: Introduction to Algebra. □

In view of the above theorem, it is evident that the 3×3 matrix A is not invertible if $\det A = 0$, since $\det \mathrm{I}_3 = 1$. It turns out that A is invertible whenever $\det A \neq 0$. But first, we define the adjugate, $\operatorname{adj} A$, of a 3×3 matrix A. Let A_{ij} be the 2×2 matrix obtained by removing the i-th row and j-th column from A. (Note that this notation conflicts with what we used earlier.) Then

$$\operatorname{adj} A := \begin{pmatrix} \det A_{11} & -\det A_{21} & \det A_{31} \\ -\det A_{12} & \det A_{22} & -\det A_{32} \\ \det A_{13} & -\det A_{23} & \det A_{33} \end{pmatrix}.$$

Theorem 8.7. For all 3×3 matrices A we have $A(\operatorname{adj} A) = (\operatorname{adj} A)A = (\det A)\mathrm{I}_3$. Thus if $\det A \neq 0$ then A is invertible, and we have $A^{-1} = \frac{1}{\det A}(\operatorname{adj} A)$.

8.3 Systems of linear equations as matrix equations

We can write any system of m simultaneous linear equations in n unknowns $x_1, x_2, \ldots, x_n$ as a matrix equation

$$A\mathbf{x} = \mathbf{d} \tag{8.1}$$

where A is an $m \times n$ matrix, $\mathbf{x}$ is an $n \times 1$ matrix and $\mathbf{d}$ is an $m \times 1$ matrix, with the entries of $A = (a_{ij})_{m\times n}$ and $\mathbf{d} = \begin{pmatrix} d_1 \\ \vdots \\ d_m \end{pmatrix}$ being known and the entries of $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ being the unknowns. The linear equations are $a_{j1}x_1 + \cdots + a_{jn}x_n = d_j$ for $1 \leqslant j \leqslant m$.

I commented previously that properly echelon form is a property of matrices. We define the matrix A to be *echelon form* if the corresponding system $A\mathbf{x} = \mathbf{d}$ of linear equations is in echelon form. So A is in echelon form (Geometry I version) if each nonzero row of A commences with strictly fewer 0s than those below it. (This forces all zero rows to occur at the 'bottom' of A.) We do not insist (for Geometry I) that the first nonzero entry in a nonzero row be 1.

If A is a square $n \times n$ matrix (so now $m = n$) such that $\det A \neq 0$ then the matrix A has a (unique) inverse A^{-1}, and by multiplying both sides of the matrix equation (8.1) by A^{-1} we see that

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{d}.$$

Since $A^{-1}A\mathbf{x} = \mathrm{I}_n\mathbf{x} = \mathbf{x}$ we deduce that

$$\mathbf{x} = A^{-1}\mathbf{d}$$

is a solution of the simultaneous equations, and indeed that it is the *unique* solution.

8.4 Determinants and inverses of $n \times n$ matrices

Notation. Throughout this section, A_{ij} will denote the $(m-1)\times(n-1)$ matrix obtained from $m \times n$ matrix A by deleting the i-th row and j-column. The notation A_{ij} only makes sense when $m, n \geqslant 1$. The (i,j)-entries of the matrices A, $\tilde{A}$ and B shall be denoted a_{ij}, $\tilde{a}_{ij}$ and b_{ij} respectively.

8.4.1 Determinants of $n \times n$ matrices

In general, if $A = (a_{ij})_{n\times n}$ is an $n \times n$ matrix the determinant of A is a sum of $n! = 1\cdot 2\cdot \cdots \cdot n$ numbers (summands), where each summand is the product of n entries of A multiplied by a sign (+1 or −1). Also, in each summand, the n entries of A used in the product come from distinct rows and columns. `<UndefinedConcepts>` Using concepts you will meet in MTH4104: Introduction to Algebra, we have

$$\det A := \sum_{\sigma\in \mathrm{S}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}, \tag{8.2}$$

where S_n denotes the set (group) of all $n!$ permutations of $\{1, 2, \ldots, n\}$ and $\operatorname{sgn}\sigma$, the *sign* or *parity* of σ, is +1 if this is even and −1 if this is odd. `</UndefinedConcepts>` From the above formula, we see that the 0×0 matrix I_0 has determinant 1, and the 1×1 matrix $\begin{pmatrix} a \end{pmatrix}$ has determinant a. (The notation $\begin{vmatrix} a \end{vmatrix}$ should not be used for the determinant of the $\begin{pmatrix} a \end{pmatrix}$ because of its confusion with the absolute value function $|a|$.) The following table summarises $n \times n$ determinants for small n.

n	#terms	determinant of A
0	1	1
1	1	a_{11}
2	2	$a_{11}a_{22} - a_{12}a_{21}$
3	6	$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$
4	24	$a_{11}a_{22}a_{33}a_{44} + \cdots$
5	120	$a_{11}a_{22}a_{33}a_{44}a_{55} + \cdots$

For $n \in \mathbb{N}$ we can define the $n \times n$ determinant recursively (that is in terms of smaller determinants) as follows. Firstly, to gives us a foundation, we define the 0×0 determinant to be 1. Then for $n \geqslant 1$ we can calculate (or even define) $\det A$ as

$$\begin{aligned}\det A &= a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13} - \cdots + (-1)^{n+1}a_{1n}\det A_{1n}\\ &= \sum_{k=1}^{n}(-1)^{k+1}a_{1k}\det A_{1k}.\end{aligned}$$

Note that this definition agrees with the formulae above for $n = 1$, 2 and 3 (as it should). Instead of expanding along the first row (as above), we can expand along (down) the first column to get

$$\det A = \sum_{k=1}^{n}(-1)^{k+1}a_{k1}\det A_{k1}.$$

In fact, we can expand along arbitrary rows and columns. For all i and j, we have

$$\det A = \sum_{k=1}^{n}(-1)^{i+k}a_{ik}\det A_{ik} = \sum_{k=1}^{n}(-1)^{k+j}a_{kj}\det A_{kj}.$$

It is possible to use the formula of (8.2) in order to show the following.

Theorem 8.8. For all $n \in \mathbb{N}$ and all $n \times n$ matrices A, we have $\det A^{\mathsf{T}} = \det A$.

Theorem 8.9. For all $n \in \mathbb{N}$, all $n \times n$ matrices A and all scalars λ, the following hold.

1. If B is obtained from A by swapping two of its rows, then $\det B = -\det A$.
2. If B is obtained from A by multiplying one of its rows by λ, then $\det B = \lambda \det A$. Thus $\det \lambda A = \lambda^n \det A$.
3. If B is obtained from A by adding λ times one of its rows to another row, then $\det B = \det A$.

Theorem 8.10. Let A, $\tilde{A}$ and B be $n \times n$ matrices differing only in the j-th row, such that $b_{jk} = a_{jk} + \tilde{a}_{jk}$ for all k. Then $\det B = \det A + \det \tilde{A}$.

In view of Theorem 8.8, Theorem 8.9 holds with the word 'row' replaced by 'column', as does Theorem 8.10 with the understanding that now $b_{kj} = a_{kj} + \tilde{a}_{kj}$ for all k. Note that if $n \geqslant 2$ and A and B are $n \times n$ matrices then $\det(A+B) \neq \det A + \det B$ in general (but not universally). Another consequence of the above theorems is that if two rows (or columns) of A are equal, or even scalar multiples of each other, then $\det A = 0$. The following is far trickier to prove.

Theorem 8.11. For all $n \in \mathbb{N}$ and all $n \times n$ matrices A and B, we have $\det(AB) = \det A \det B$.

8.4.2 Adjugates and inverses of $n \times n$ matrices

The 0×0 matrix I_0 has inverse I_0. From now on we let $n \geqslant 1$. If $A = \begin{pmatrix} a \end{pmatrix}$ then $\operatorname{adj} A = \begin{pmatrix} 1 \end{pmatrix}$, and if $a = \det A \neq 0$ then A is invertible and $A^{-1} = \begin{pmatrix} \frac{1}{a} \end{pmatrix}$, and if $a = \det A = 0$ then A is not invertible. We have already seen the adjugate and inverse of a 2×2 and a 3×3 matrix.

In general, we let $A = (a_{ij})_{n \times n}$ for some $n \geqslant 1$, and let A_{ij} be the result of removing the i-th row and j-th column from A. The (i, j)-*minor* of A is $m_{ij} := \det A_{ij}$. The *cofactor* matrix of A is $C = (c_{ij})_{n \times n}$ where $c_{ij} := (-1)^{i+j} m_{ij} = (-1)^{i+j} \det A_{ij}$. The *adjugate* of A is the transpose of the cofactor matrix. That is $\operatorname{adj} A := C^{\mathsf{T}}$, and so $\operatorname{adj} A$ has (i, j)-entry $(-1)^{i+j} \det A_{ji}$.

The following two theorems hold, and show one way (not always very efficient) to compute A^{-1} in the case when A is invertible.

Theorem 8.12. For all $n \geqslant 1$ we have $A(\operatorname{adj} A) = (\operatorname{adj} A)A = (\det A)\mathrm{I}_n$ for all $n \times n$ matrices A.

Theorem 8.13. For all $n \geqslant 1$, the $n \times n$ matrix A is invertible if and only if $\det A \neq 0$, and if $\det A \neq 0$ we have $A^{-1} = \frac{1}{\det A}(\operatorname{adj} A)$.