Chapter 6

The Vector Product

6.1 Parallel vectors

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors. We say that $\mathbf{u}$ and $\mathbf{v}$ are *parallel*, and write $\mathbf{u} \parallel \mathbf{v}$, if $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$ (which will also force $\mathbf{v}$ to be a scalar multiple of $\mathbf{u}$). Note that $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if they have the same or opposite directions, which happens exactly when $\mathbf{u}$ and $\mathbf{v}$ are at an angle of 0 or π.

Example. The vectors $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix}$ are parallel, but $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix}$ are not parallel.

The relation of parallelism has the following properties, where $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are *nonzero* vectors.

1. $\mathbf{u} \parallel \mathbf{u}$ for all vectors $\mathbf{u}$. [This property is called *reflexivity*.]

2. $\mathbf{u} \parallel \mathbf{v}$ implies that $\mathbf{v} \parallel \mathbf{u}$. [This property is called *symmetry*.]

3. $\mathbf{u} \parallel \mathbf{v}$ and $\mathbf{v} \parallel \mathbf{w}$ implies that $\mathbf{u} \parallel \mathbf{w}$. [This property is called *transitivity*.]

A relation that is reflexive, symmetric and transitive is called an *equivalence relation*. This concept was introduced in MTH4110: Mathematical Structures. Thus, parallelism is a equivalence relation on the set of nonzero vectors

Note. A wording like "$\mathbf{u}$ and $\mathbf{v}$ are parallel if ..." presumes that the property of parallelism is symmetric between $\mathbf{u}$ and $\mathbf{v}$. This may be obvious from the definition that follows, or else a proof would need to be supplied. If you want to define a concept that is asymmetric (or not obviously symmetric), a wording like "$\mathbf{u}$ is parallel to $\mathbf{v}$ if ..." is more appropriate. [Can you prove that the relations of parallelism and collinearity (see below) are symmetric? Be careful of the zero vector for the latter relation.]

Also, when one provides more than one definition of a concept, one should prove the equivalence of the definitions. Can you prove the various definitions of collinear to be equivalent?

6.2 Collinear vectors

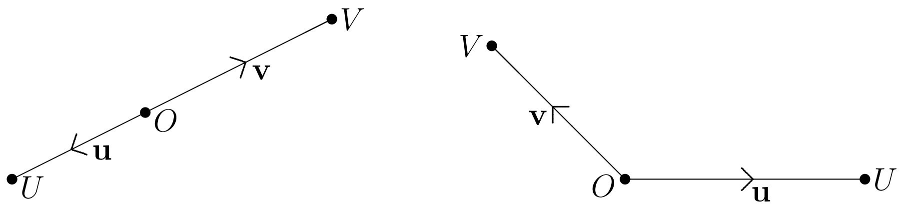
It is useful to extend the notion of parallelism to pairs of vectors involving the zero vector. However, we shall give this notion a different name, and call it collinearity. We say that two vectors $\mathbf{u}$ and $\mathbf{v}$ are *collinear* if $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ or $\mathbf{u}$ and $\mathbf{v}$ are parallel (this includes the case $\mathbf{u} = \mathbf{v} = \mathbf{0}$).

An equivalent definition of collinearity is that $\mathbf{u}$ and $\mathbf{v}$ are collinear if there exist [real] numbers (scalars) α and β not both zero such that

$$\alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{0}. \tag{6.1}$$

(This covers the case when one or both of $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{0}$, as well as the general case when $\mathbf{u}$ and $\mathbf{v}$ are parallel.) The relation of collinearity is reflexive and symmetric (in all dimensions), but is not transitive in dimension at least 2. However, it is nearly transitive [not a technical term]: if $\mathbf{u}$ and $\mathbf{v}$ are collinear, and $\mathbf{v}$ and $\mathbf{w}$ are collinear, then either $\mathbf{u}$ and $\mathbf{w}$ are collinear or $\mathbf{v} = \mathbf{0}$ (or both).

Another definition of collinearity is that $\mathbf{u}$ and $\mathbf{v}$ are collinear if O, U and V all lie on some [straight] line, where $\overrightarrow{OU}$ represents $\mathbf{u}$ and $\overrightarrow{OV}$ represents $\mathbf{v}$. This line is unique except in the case $\mathbf{u} = \mathbf{v} = \mathbf{0}$, that is $O = U = V$. Two diagrams illustrating this concept are given below. In the left-hand one $\mathbf{u}$ and $\mathbf{v}$ are collinear, and in the right-hand one they are not.



Note. The word collinear is logically made up of two parts: the prefix *co-*, and the stem *linear*. Thus it would seem that the logical spelling of *collinear* should be as *colinear*. However, *co-* is a variant of *com-* or *con-*, roughly meaning 'with' or 'together', and this assimilates to *col-* and *cor-* before words beginning with L and R respectively. Thus, in contrast to words like *coplanar*, we get words like *collinear* and *correlation* since the plain *co-* prefix is rare before words beginning with L and R. (The phenomenon of assimilation also affects the prefix Latinate prefix *in-*, giving us words like *impossible*, *immaterial*, *illegal* and *irrational*. The Germanic prefix *un-* is unaffected by this phenonemon.)

6.3 Coplanar vectors

We say that vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are *coplanar* if the points O, U, V, W all lie on some plane, where $\overrightarrow{OU}$, $\overrightarrow{OV}$, $\overrightarrow{OW}$ represent the free vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ respectively. (Note that this definition is symmetric in $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.) Thus we see that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are coplanar if and only if either $\mathbf{u}$ and $\mathbf{v}$ are collinear or there exist scalars λ and μ such that $\mathbf{w} = \lambda\mathbf{u}+\mu\mathbf{v}$.

From the above, we get the following symmetrical algebraic formulation of coplanarity: $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are *coplanar* if there exist [real] numbers (scalars) α, β, γ such that

$$\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} = \mathbf{0} \quad \text{and} \quad (\alpha, \beta, \gamma) \neq (0, 0, 0). \tag{6.2}$$

Note that to determine whether a particular triple $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is coplanar, we can use the above equation $\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} = \mathbf{0}$, which induces 3 linear equations in the unknowns α, β, γ (or n equations if $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ rather than $\mathbb{R}^3$). These equations always have the solution $\alpha = \beta = \gamma = 0$, and $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are coplanar if and only if the equations have a solution *other than* $\alpha = \beta = \gamma = 0$.

Example 1. If $\mathbf{u} = \mathbf{0}$, $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$ then $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are coplanar. For example, if $\mathbf{u} = \mathbf{0}$, we can take $\alpha = 1$, $\beta = \gamma = 0$ in Equation 6.2. Geometrically, the points O, U, V, W consist of at most 3 distinct points, and any three points (in $\mathbb{R}^3$) lie on at least one plane.

Example 2. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are (nonzero and) parallel. Then $\mathbf{v} = \lambda\mathbf{u}$ for some scalar λ. Therefore we have $\lambda\mathbf{u} - \mathbf{v} = \mathbf{0}$, and so in Equation 6.2 we can take $\alpha = \lambda$, $\beta = -1$, $\gamma = 0$. Geometrically, O, U, V lie on one line (which is unique since $U, V \neq O$), and so O, U, V, W lie on at least one plane (which is unique, except when W lies on the line determined by O, U, V).

Example 3. We let $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$. Then Equation 6.2 yields the following linear equations in α, β, γ:

$$\left.\begin{aligned} \alpha + 4\beta + 7\gamma &= 0 \\ 2\alpha + 5\beta + 8\gamma &= 0 \\ 3\alpha + 6\beta + 9\gamma &= 0 \end{aligned}\right\}.$$

Echelonisation gives the following system of equations:

$$\left.\begin{aligned} \alpha + 4\beta + 7\gamma &= 0 \\ -\,3\beta - 6\gamma &= 0 \\ 0 &= 0 \end{aligned}\right\}.$$

Back substitution then gives the solution $\alpha = t$, $\beta = -2t$, $\gamma = t$, where t can be any real number. Setting $t = 1$ (any nonzero value will do), we see that $\mathbf{u} - 2\mathbf{v} + \mathbf{w} = \mathbf{0}$, so that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are coplanar.

Example 4. We let $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$. Then Equation 6.2 yields the following linear equations in α, β, γ:

$$\left.\begin{aligned} \alpha + 2\beta + 3\gamma &= 0 \\ 2\alpha + 3\beta + \gamma &= 0 \\ 3\alpha + \beta + 2\gamma &= 0 \end{aligned}\right\}.$$

Echelonisation gives the following system of equations:

$$\left.\begin{aligned} \alpha + 2\beta + 3\gamma &= 0 \\ -\beta - 5\gamma &= 0 \\ 18\gamma &= 0 \end{aligned}\right\}.$$

Thus we see that the only solution to this system of equations is $\alpha = \beta = \gamma = 0$. Therefore **u**, **v**, **w** are *not* coplanar.

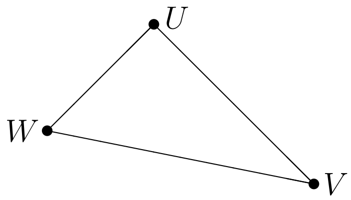
6.4 Right-handed and left-handed triples

Suppose now that **u**, **v**, **w** are not coplanar. Pick an origin O in 3-space, and define U, V, W by the condition that $\overrightarrow{OU}$, $\overrightarrow{OV}$, $\overrightarrow{OW}$ represent **u**, **v**, **w** respectively. We now give three methods to determine whether we have a right-handed or left-handed triple. You should try to convince yourselves of the equivalence of these methods. Some of the results stated under the "finger exercises" (Section 6.4.1) are easier to prove with one of these methods than another.

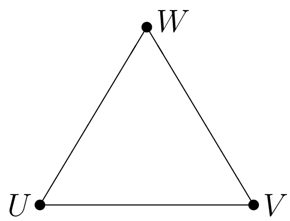
Method 1. We let the plane determined by U, V, W be the page of these notes. (Note that U, V, W are not all on the same line, so that this plane is unique.) We now orient ourselves so that O is in front of the page, which is the same side as us. (Note that O cannot be on the page.) We now follow the vertices of the triangle UVW around *clockwise*. If they occur in the order U, V, W (or V, W, U or W, U, V) then the triple is right-handed. Else, for the orders U, W, V and V, U, W and W, V, U, the triple is left-handed.

If O is behind the page (the other side to us), then the *anticlockwise* orders U, V, W and V, W, U and W, U, V give us a right-handed triple, while the anticlockwise orders U, W, V and V, U, W and W, V, U give us a left-handed triple.

The following diagrams illustrate the situations that arise, where the page is the plane containing U, V and W. Note that O being on the page would make **u**, **v**, **w** coplanar, so this does not happen here.



right-handed if O is **in front of** the page
left-handed if O is **behind** the page



right-handed if O is **behind** the page
left-handed if O is **in front of** the page

Method 2. We let the plane determined by O, U, V be the page of these notes. We then look at the plane from the side such the angle from $\mathbf{u}$ to $\mathbf{v}$ *proceeding anticlockwise* is between 0 and π. (If the anticlockwise angle lies between π and 2π then look at the plane from the other side.) If $\mathbf{w}$ points 'towards' you, that is W is the same side of the O, U, V-plane as you are, then $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is right-handed. If $\mathbf{w}$ points 'away from' you then $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is left-handed.

Method 3. Using your right hand put your thumb in the direction of $\mathbf{u}$, and your first [index] finger in the direction of $\mathbf{v}$. If W lies on the side of the plane through O, U, V indicated by your second [middle] finger, we call $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ a *right-handed triple*; otherwise it is a *left-handed triple*.

For any triple $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ of vectors, precisely one of the following properties holds: it is coplanar, it is right-handed, or it is left-handed.

Examples. $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ is a right-handed triple. $\mathbf{i}$, $\mathbf{j}$, $-\mathbf{k}$ and $\mathbf{k}$, $\mathbf{j}$, $\mathbf{i}$ are both left-handed triples.

Note. The vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ forming a right-handed or left-handed triple need not be mutually orthogonal, but they *must not* be coplanar.

6.4.1 Some finger exercises (for you to do)

If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are coplanar, then so is any triple that is a permutation of $\pm\mathbf{u}$, $\pm\mathbf{v}$, $\pm\mathbf{w}$, for any of the 8 possibilities for sign. From now on, we assume that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are not coplanar. The basic operations we can do to a triple of vectors are:

1. multiply one the vectors by a constant $\lambda > 0$;

2. negate one of them; or

3. swap two of them.

The first operation preserves the handedness of a triple, but the latter two operations send right-handed triples to left-handed triples and vice versa. (Each of these operations sends coplanar triples to coplanar triples.) Combining these operations, we see that negating an even number of the vectors preserves handedness, while negating an odd number of the vectors changes the handedness. One can convert $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ to each of $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}$ and $\mathbf{w}$, $\mathbf{u}$, $\mathbf{v}$ using two swaps. Thus one sees that (in 3 dimensions) cycling $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ does not change the handedness of the system. We have the following.

- If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is a right-handed triple, then the following triples are also right-handed: $\mathbf{u}, \mathbf{v}, \mathbf{w}$; $\mathbf{v}, \mathbf{w}, \mathbf{u}$; $\mathbf{w}, \mathbf{u}, \mathbf{v}$; $\mathbf{u}, -\mathbf{v}, -\mathbf{w}$; $-\mathbf{u}, \mathbf{v}, -\mathbf{w}$; $-\mathbf{u}, -\mathbf{v}, \mathbf{w}$; $\mathbf{u}, -\mathbf{w}, \mathbf{v}$; $\mathbf{u}, \mathbf{w}, -\mathbf{v}$; $-\mathbf{u}, \mathbf{w}, \mathbf{v}$; $\mathbf{w}, \mathbf{v}, -\mathbf{u}$; $\mathbf{w}, -\mathbf{v}, \mathbf{u}$; $-\mathbf{w}, \mathbf{v}, \mathbf{u}$; $-\mathbf{w}, -\mathbf{v}, -\mathbf{u}$; and so on.

- If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is a right-handed triple, then the following triples are left-handed: $\mathbf{u}, \mathbf{w}, \mathbf{v}$; $\mathbf{w}, \mathbf{v}, \mathbf{u}$; $\mathbf{v}, \mathbf{u}, \mathbf{w}$; $-\mathbf{u}, \mathbf{v}, \mathbf{w}$; $\mathbf{u}, -\mathbf{v}, \mathbf{w}$; $\mathbf{u}, \mathbf{v}, -\mathbf{w}$; $-\mathbf{u}, -\mathbf{v}, -\mathbf{w}$; $-\mathbf{v}, -\mathbf{w}, -\mathbf{u}$; $-\mathbf{w}, -\mathbf{u}, -\mathbf{v}$; $\mathbf{w}, -\mathbf{v}, -\mathbf{u}$; $-\mathbf{w}, \mathbf{v}, -\mathbf{u}$; $-\mathbf{w}, -\mathbf{v}, \mathbf{u}$; and so on.

6.5 The vector product

We now describe a method of multiplying two vectors to obtain another vector.

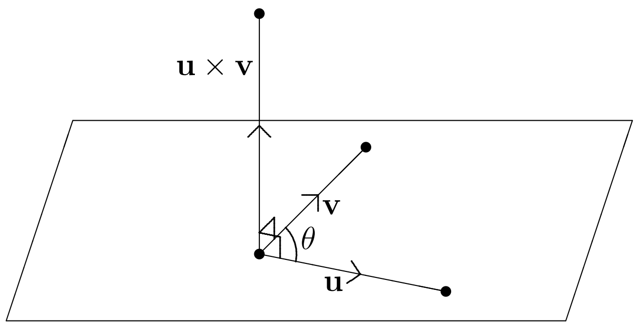
Definition 6.1. Suppose that $\mathbf{u}$, $\mathbf{v}$ are nonzero non-parallel vectors (of $\mathbb{R}^3$) at angle θ. Then the *vector product* or *cross product* $\mathbf{u} \times \mathbf{v}$ is defined to be the vector satisfying:

(i) $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$ (note that $\sin\theta > 0$ here, since $0 < \theta < \pi$);

(ii) $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$; and

(iii) $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} \times \mathbf{v}$ is a right-handed triple.

If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ or $\mathbf{u}$, $\mathbf{v}$ are parallel, then the vector product $\mathbf{u} \times \mathbf{v}$ is defined to be $\mathbf{0}$. Thus $|\mathbf{u} \times \mathbf{v}| = 0 = |\mathbf{u}||\mathbf{v}| \sin\theta$ also holds when $\mathbf{u}$ and $\mathbf{v}$ are (nonzero and) parallel. A picture illustrating a typical cross product appears below.



Note. Occasionally, you will see the cross product $\mathbf{u} \times \mathbf{v}$ written as $\mathbf{u} \wedge \mathbf{v}$. However, this wedge product properly means something else, so you should not use such a notation.

We note that if $\mathbf{u}$, $\mathbf{v}$ are nonzero and non-parallel, then $|\mathbf{u} \times \mathbf{v}| > 0$ (so that $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$); otherwise $|\mathbf{u} \times \mathbf{v}| = 0$ (so that $\mathbf{u} \times \mathbf{v} = \mathbf{0}$). In particular, for all vectors $\mathbf{u}$, we have $\mathbf{u} \times \mathbf{u} = \mathbf{0}$. Further, we note that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, since:

(i) $|\mathbf{k}| = 1 = 1 \times 1 \times 1 = |\mathbf{i}||\mathbf{j}| \sin\frac{\pi}{2}$;

(ii) $\mathbf{k}$ is orthogonal to both $\mathbf{i}$ and $\mathbf{j}$; and

(iii) $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ is a right-handed triple.

Similarly, we get the following table of cross products:

	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{0}$	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$-\mathbf{k}$	$\mathbf{0}$	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	$\mathbf{0}$

where the entry in the $\mathbf{u}$ row and $\mathbf{v}$ column is $\mathbf{u} \times \mathbf{v}$.

6.6 Properties of the vector product

For all vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and for all scalars α, the following properties hold

1. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$;
2. $(\alpha\mathbf{u}) \times \mathbf{v} = \alpha(\mathbf{u} \times \mathbf{v})$ and $\mathbf{u} \times (\alpha\mathbf{v}) = \alpha(\mathbf{u} \times \mathbf{v})$;
3. $\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w}) = \mathbf{v}\cdot(\mathbf{w} \times \mathbf{u}) = \mathbf{w}\cdot(\mathbf{u} \times \mathbf{v})$ and
   $\mathbf{u}\cdot(\mathbf{w} \times \mathbf{v}) = \mathbf{w}\cdot(\mathbf{v} \times \mathbf{u}) = \mathbf{v}\cdot(\mathbf{u} \times \mathbf{w}) = -[\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w})]$;
4. $\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v})\cdot\mathbf{w}$;
5. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$ and $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{v} \times \mathbf{w})$.

The proofs of these will be done below, with some other necessary results interspersed. We also note the result that the vector product is not associative, that is in general $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$, even though both sides of this are always defined. We leave it as an exercise to find an example of this non-equality. There are, however, triples $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ of vectors such that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$; for example, we have $\mathbf{i} \times (\mathbf{j} \times \mathbf{k}) = \mathbf{i} \times \mathbf{i} = \mathbf{0} = \mathbf{k} \times \mathbf{k} = (\mathbf{i} \times \mathbf{j}) \times \mathbf{k}$. Nor is the vector product commutative.

In the proofs below, we shall make use of certain facts without noting them explictly. One such fact is that collinearity is a symmetric relation. That is, $\mathbf{u}$ and $\mathbf{v}$ are collinear if and only if $\mathbf{v}$ and $\mathbf{u}$ are collinear.

Theorem 6.2. For all vectors $\mathbf{u}$ and $\mathbf{v}$ we have $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$. This property is called *anti-commutativity.*

Note. The fact that the vector product is anti-commutative ***does not*** prove that it fails to be commutative. It could be that $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ for all $\mathbf{u}$ and $\mathbf{v}$, or that $-(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times \mathbf{v}$ even when $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$. You must exhibit an ***explicit example*** to show that the vector product is not commutative, for example $\mathbf{i} \times \mathbf{j} = \mathbf{k} \neq -\mathbf{k} = \mathbf{j} \times \mathbf{i}$. (In some situations there may be more subtle ways to show a property does not always hold, but the explicit example still seems to be the best approach.)

Proof. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ or $\mathbf{u}$ and $\mathbf{v}$ are parallel (that is if $\mathbf{u}$ and $\mathbf{v}$ are collinear) then $\mathbf{v} \times \mathbf{u} = \mathbf{0} = -\mathbf{0} = -(\mathbf{u} \times \mathbf{v})$. Otherwise, we let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$, and let $\mathbf{w} = \mathbf{u} \times \mathbf{v}$. Now $|-\mathbf{w}| = |\mathbf{w}| = |\mathbf{u}||\mathbf{v}| \sin\theta = |\mathbf{v}||\mathbf{u}| \sin\theta$, $-\mathbf{w}$ is orthogonal to $\mathbf{u}$, $\mathbf{v}$ (since $\mathbf{w}$ is), and $\mathbf{v}$, $\mathbf{u}$, $-\mathbf{w}$ is a right-handed triple. Thus $\mathbf{v} \times \mathbf{u} = -\mathbf{w} = -(\mathbf{u} \times \mathbf{v})$. $\square$

Theorem 6.3. For all vectors $\mathbf{u}$ and $\mathbf{v}$ and all scalars α we have $(\alpha\mathbf{u}) \times \mathbf{v} = \alpha(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (\alpha\mathbf{v})$.

Proof. We prove the first of these of equalities only. The proof of the other is similar and is left as an exercise (on Coursework 5). Alternatively, we can use properties we will have already established, for we have

$$\mathbf{u} \times (\alpha\mathbf{v}) = -((\alpha\mathbf{v}) \times \mathbf{u}) = -(\alpha(\mathbf{v} \times \mathbf{u})) = -(\alpha(-(\mathbf{u} \times \mathbf{v}))) = \alpha(\mathbf{u} \times \mathbf{v}).$$

If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ or $\mathbf{u}$, $\mathbf{v}$ are parallel or $\alpha = 0$ then $(\alpha\mathbf{u}) \times \mathbf{v} = \mathbf{0} = \alpha(\mathbf{u} \times \mathbf{v})$. Otherwise (when $\alpha \neq 0$ and $\mathbf{u}, \mathbf{v}$ not collinear), we let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$, and let $\mathbf{w} = \mathbf{u} \times \mathbf{v}$. The angle between $\alpha\mathbf{u}$ and $\mathbf{v}$ is θ if $\alpha > 0$ and $\pi - \theta$ if $\alpha < 0$, and we have

$$|\alpha\mathbf{w}| = |\alpha||\mathbf{w}| = |\alpha||\mathbf{u}||\mathbf{v}| \sin\theta = |\alpha\mathbf{u}||\mathbf{v}| \sin\theta = |\alpha\mathbf{u}||\mathbf{v}| \sin(\pi - \theta).$$

Moreover, $\alpha\mathbf{w}$ is orthogonal to $\alpha\mathbf{u}$ and $\mathbf{v}$ (since $\mathbf{w}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$), and $\alpha\mathbf{u}$, $\mathbf{v}$, $\alpha\mathbf{w}$ is a right-handed triple (whether $\alpha > 0$ or $\alpha < 0$). Thus $(\alpha\mathbf{u}) \times \mathbf{v} = \alpha(\mathbf{u} \times \mathbf{v})$. $\square$

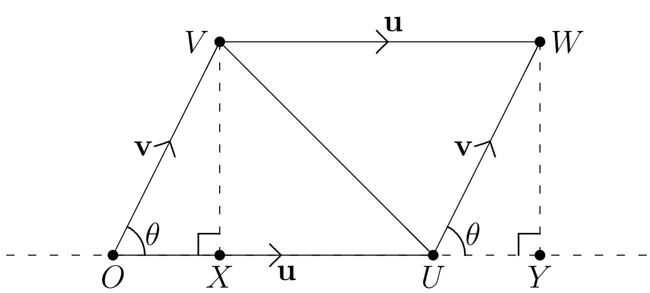
6.6.1 The area of a parallelogram and triangle

In this subsection, we prove formulae involving the vector product for areas of parallelograms and triangles having sides that represent the vectors $\mathbf{u}$ and $\mathbf{v}$.

Definition. Two geometric figures are said to be *congruent* if one can be obtained from the other by a combination of rotations, reflexions and translations. A figure is always congruent to itself, and two congruent figures always have the same area (if indeed they have an area at all: some really really weird bounded figures do not have an area). Two geometric figures are said to be *similar* if one can be obtained from the other by a combination of rotations, reflexions, translations, and scalings by nonzero amounts. Both congruence and similarity are equivalence relations on the set of geometric figures.

Theorem 6.4. Let $\overrightarrow{OU}$ and $\overrightarrow{OV}$ represent $\mathbf{u}$ and $\mathbf{v}$ respectively, and let W be the point making $OUWV$ into a parallelogram. Then the parallelogram $OUWV$ has area $|\mathbf{u} \times \mathbf{v}|$ and the triangle OUV has area $\frac{1}{2}|\mathbf{u} \times \mathbf{v}|$.

Proof. If $\mathbf{u} = \mathbf{0}$, $\mathbf{v} = \mathbf{0}$ or $\mathbf{u}$ and $\mathbf{v}$ are parallel then $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ and O, U, V, W all lie on the same line, so that the (degenerate) parallelogram $OUWV$ and triangle OUV both have area $0 = |\mathbf{u} \times \mathbf{v}| = \frac{1}{2}|\mathbf{u} \times \mathbf{v}|$. For the rest of the proof, we assume that $\mathbf{u}$ and $\mathbf{v}$ are not collinear, and for clarity, we also refer to the diagram below.



Otherwise, let ℓ be the line through O and U, and let X and Y be the points on ℓ with VX and WY perpendicular to ℓ. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Now triangles OXV and UYW are congruent, and thus have the same area, and so $OUWV$ has the same area as the rectangle $XYWV$, which is:

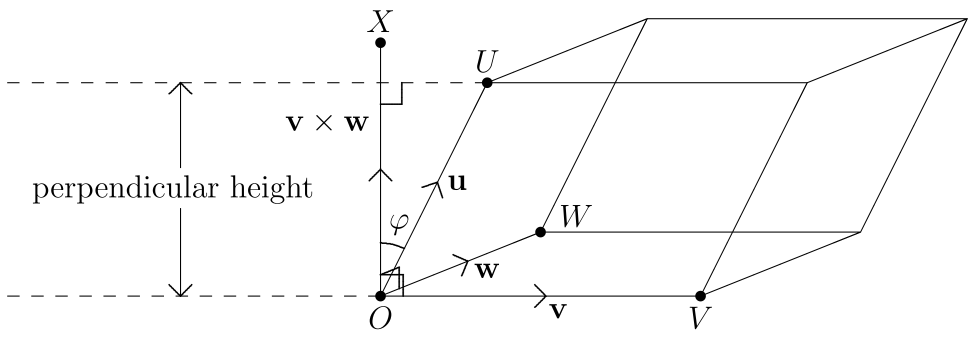
$$|\overrightarrow{VW}||\overrightarrow{VX}| = |\mathbf{u}|(|\mathbf{v}| \sin\theta) = |\mathbf{u} \times \mathbf{v}|.$$

The triangles OUV and WUV are congruent triangles covering the whole of the parallelogram $OUWV$ without overlap (except on the line UV of area 0). Thus OUV has half the area of $OUWV$, that is $\frac{1}{2}|\mathbf{u} \times \mathbf{v}|$. □

6.6.2 The triple scalar product

Definition. The *triple scalar product* of the ordered triple of vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$. Note that we have not defined the cross product of a scalar and a vector (either way round), and so $\mathbf{u} \times (\mathbf{v}\cdot\mathbf{w})$ and $(\mathbf{u}\cdot\mathbf{v}) \times \mathbf{w}$ do not exist.

Theorem 6.5. The volume of a *parallelepiped* with sides corresponding to $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ (as per the diagram below) is $|\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w})|$.



Proof. Each face of a parallelepiped is a parallelogram, and here the base is a parallelogram with sides corresponding to $\mathbf{v}$ and $\mathbf{w}$. Thus the parallelepiped has volume

$$V = \text{area of base} \times \text{perpendicular height} = |\mathbf{v} \times \mathbf{w}| \times \text{perpendicular height},$$

by Theorem 6.4. If $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ or $\mathbf{u} = \mathbf{0}$ then $V = 0 = |\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w})|$. Otherwise, the perpendicular height is $||\mathbf{u}| \cos\varphi| = |\mathbf{u}||\cos\varphi|$ where φ is the angle between $\mathbf{u}$ and $\mathbf{v}\times\mathbf{w}$, since $\mathbf{v} \times \mathbf{w}$ is orthogonal to the base. Thus

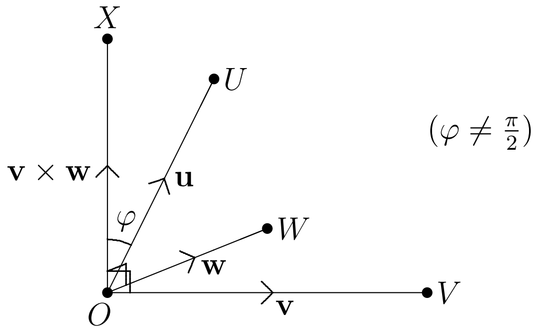
$$V = |\mathbf{v} \times \mathbf{w}||\mathbf{u}||\cos\varphi| = ||\mathbf{u}||\mathbf{v} \times \mathbf{w}| \cos\varphi| = |\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w})|,$$

by the definition of the scalar product. □

Remark. The volume of the tetrahedron determined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is $\frac{1}{6}|\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w})|$. The volume is calculated as $\frac{1}{3} \times$ base area $\times$ perpendicular height. The tetrahedron has four vertices, namely O, U, V and W, and four triangular faces, which are OUV, OVW, OWU and UVW (see diagram above).

When is a triple scalar product 0? positive? negative?

We have $\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w}) = 0$ if and only if the volume of a parallelepiped with sides corresponding to $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is 0, which happens exactly when $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are coplanar. Otherwise, we consider the following diagram (on the next page), where $\varphi \neq \frac{\pi}{2}$, since we do not wish to have U in the plane determined by O, V and W.



Now $\mathbf{v} \times \mathbf{w}$ is orthogonal to the plane Π through O, V and W, and $\mathbf{v}$, $\mathbf{w}$, $\mathbf{v} \times \mathbf{w}$ is a right-handed triple. Also, $\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w}) = |\mathbf{u}||\mathbf{v} \times \mathbf{w}| \cos\varphi$, where φ is the angle between $\mathbf{u}$ and $\mathbf{v} \times \mathbf{w}$.

If $\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w}) > 0$ then $0 \leqslant \varphi < \frac{\pi}{2}$, and so U is on the same side of Π as X. This implies that $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}$ is a right-handed triple, and thus so is $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

If $\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w}) < 0$ then $\frac{\pi}{2} < \varphi \leqslant \pi$, and so U is on the other side of Π to X. This implies that $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}$ is a left-handed triple, and thus so is $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.
Thus we conclude the following.

1. $\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w}) = 0$ if and only if $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are coplanar.

2. $\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w}) > 0$ if and only if $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is a right-handed triple.

3. $\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w}) < 0$ if and only if $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is a left-handed triple.

Theorem 6.6. For all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ we have $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{v}\cdot(\mathbf{w}\times\mathbf{u}) = \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})$. Thus $\mathbf{u}\cdot(\mathbf{w} \times \mathbf{v}) = \mathbf{w}\cdot(\mathbf{v} \times \mathbf{u}) = \mathbf{v}\cdot(\mathbf{u} \times \mathbf{w}) = -[\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w})]$ for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

Proof. We prove the first line. The second line follows from the first, together with anti-commutativity of the cross product and various properties of the dot product.

In absolute terms, each of these triple products ($\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$, $\mathbf{v}\cdot(\mathbf{w}\times\mathbf{u})$ and $\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})$) gives the volume V of a parallelepiped with sides corresponding to $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are coplanar then each triple product gives $V = 0$.

If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is a right-handed triple, then so are $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}$ and $\mathbf{w}$, $\mathbf{u}$, $\mathbf{v}$, and each triple product gives $+V$.

If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is a left-handed triple, then so are $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}$ and $\mathbf{w}$, $\mathbf{u}$, $\mathbf{v}$, and each triple product gives $-V$. □

Theorem 6.7. For all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ we have $\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v})\cdot\mathbf{w}$.

Proof. We have $(\mathbf{u} \times \mathbf{v})\cdot\mathbf{w} = \mathbf{w}\cdot(\mathbf{u} \times \mathbf{v}) = \mathbf{u}\cdot(\mathbf{v} \times \mathbf{w})$, using commutativity of the dot product followed by the previous theorem. □

6.6.3 The distributive laws for the vector product

Theorem 6.8. For all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ we have $(\mathbf{u}+\mathbf{v})\times\mathbf{w} = (\mathbf{u}\times\mathbf{w})+(\mathbf{v}\times\mathbf{w})$ and $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = (\mathbf{u}\times\mathbf{v})+(\mathbf{u}\times\mathbf{w})$.

Proof. We prove just the first of these. Let $\mathbf{t}$ be any vector. Then we have

$$\begin{aligned}
\mathbf{t}\cdot((\mathbf{u}+\mathbf{v})\times\mathbf{w}) &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{w}\times\mathbf{t}) && \text{(by Theorem 6.6)}\\
&= \mathbf{u}\cdot(\mathbf{w}\times\mathbf{t})+\mathbf{v}\cdot(\mathbf{w}\times\mathbf{t}) && \text{(by Distributive Law for }\cdot)\\
&= \mathbf{t}\cdot(\mathbf{u}\times\mathbf{w})+\mathbf{t}\cdot(\mathbf{v}\times\mathbf{w}) && \text{(by Theorem 6.6)}
\end{aligned}$$

Therefore we have

$$\begin{aligned}
0 &= \mathbf{t}\cdot((\mathbf{u}+\mathbf{v})\times\mathbf{w}) - \mathbf{t}\cdot(\mathbf{u}\times\mathbf{w}) - \mathbf{t}\cdot(\mathbf{v}\times\mathbf{w})\\
&= \mathbf{t}\cdot(((\mathbf{u}+\mathbf{v})\times\mathbf{w}) - (\mathbf{u}\times\mathbf{w}) - (\mathbf{v}\times\mathbf{w}))
\end{aligned}$$

Let $\mathbf{s} = ((\mathbf{u}+\mathbf{v})\times\mathbf{w})-(\mathbf{u}\times\mathbf{w})-(\mathbf{v}\times\mathbf{w})$. Since $\mathbf{t}$ can be any vector we have $\mathbf{s}\cdot\mathbf{t} = \mathbf{t}\cdot\mathbf{s} = 0$ for every vector $\mathbf{t}$, and so by the Feedback Question (Part (c)) of Coursework 2, we must have $\mathbf{s} = \mathbf{0}$. Thus $(\mathbf{u}+\mathbf{v})\times\mathbf{w} = (\mathbf{u}\times\mathbf{w})+(\mathbf{v}\times\mathbf{w})$, as required.

The proof of the second equality has a similar proof to the first, or it can be deduced from the first using the anti-commutative property of the vector product. □

6.7 The vector product in coördinates

We now use the rules for the vector product we have proved to find a formula for the vector product of two vectors given in coördinates.

Theorem 6.9. Let $\mathbf{u} = \begin{pmatrix} u_1\\ u_2\\ u_3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1\\ v_2\\ v_3 \end{pmatrix}$. Then we have:

$$\mathbf{u}\times\mathbf{v} = \begin{pmatrix} u_1\\ u_2\\ u_3 \end{pmatrix}\times\begin{pmatrix} v_1\\ v_2\\ v_3 \end{pmatrix} = \begin{pmatrix} u_2v_3-u_3v_2\\ u_3v_1-u_1v_3\\ u_1v_2-u_2v_1 \end{pmatrix}.$$

Proof. We have $\mathbf{u} = u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$. Therefore, the distributive and scalar multiplication laws (Theorems 6.3 and 6.8) give us:

$$\begin{aligned}
\mathbf{u}\times\mathbf{v} &= (u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k})\times(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k})\\
&= (u_1\mathbf{i}\times v_1\mathbf{i})+(u_1\mathbf{i}\times v_2\mathbf{j})+(u_1\mathbf{i}\times v_3\mathbf{k})+(u_2\mathbf{j}\times v_1\mathbf{i})\\
&\qquad +(u_2\mathbf{j}\times v_2\mathbf{j})+(u_2\mathbf{j}\times v_3\mathbf{k})+(u_3\mathbf{k}\times v_1\mathbf{i})+(u_3\mathbf{k}\times v_2\mathbf{j})+(u_3\mathbf{k}\times v_3\mathbf{k})\\
&= u_1v_1(\mathbf{i}\times\mathbf{i})+u_1v_2(\mathbf{i}\times\mathbf{j})+u_1v_3(\mathbf{i}\times\mathbf{k})+u_2v_1(\mathbf{j}\times\mathbf{i})\\
&\qquad +u_2v_2(\mathbf{j}\times\mathbf{j})+u_2v_3(\mathbf{j}\times\mathbf{k})+u_3v_1(\mathbf{k}\times\mathbf{i})+u_3v_2(\mathbf{k}\times\mathbf{j})+u_3v_3(\mathbf{k}\times\mathbf{k})\\
&= \mathbf{0}+u_1v_2\mathbf{k}+u_1v_3(-\mathbf{j})+u_2v_1(-\mathbf{k})+\mathbf{0}+u_2v_3\mathbf{i}+u_3v_1\mathbf{j}+u_3v_2(-\mathbf{i})+\mathbf{0}\\
&= (u_2v_3-u_3v_2)\mathbf{i}+(-u_1v_3+u_3v_1)\mathbf{j}+(u_1v_2-u_2v_1)\mathbf{k}\\
&= (u_2v_3-u_3v_2)\mathbf{i}-(u_1v_3-u_3v_1)\mathbf{j}+(u_1v_2-u_2v_1)\mathbf{k},
\end{aligned}$$

which is the result we wanted. □

There is a more useful way to remember this formula for $\mathbf{u} \times \mathbf{v}$. We define:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc,$$

which is the *determinant* of the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$. Then we have:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \mathbf{i} + \begin{vmatrix} u_3 & v_3 \\ u_1 & v_1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \mathbf{k} = \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \mathbf{k}.$$

Example. Let $\mathbf{u} = \begin{pmatrix} 2 \\ -5 \\ 7 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -3 \\ -1 \\ 4 \end{pmatrix}$. Then

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} -5 & -1 \\ 7 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & -3 \\ 7 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -3 \\ -5 & -1 \end{vmatrix} \mathbf{k} \\ &= (-20 - (-7))\mathbf{i} - (8 - (-21))\mathbf{j} + (-2 - 15)\mathbf{k} = -13\mathbf{i} - 29\mathbf{j} - 17\mathbf{k}. \end{aligned}$$

In order to *check* (not *prove*) that we have calculated $\mathbf{u} \times \mathbf{v}$ correctly, we evaluate the scalar products $\mathbf{u} \cdot \mathbf{w}$ and $\mathbf{v} \cdot \mathbf{w}$, where we have calculated $\mathbf{u} \times \mathbf{v}$ to be $\mathbf{w}$. Both scalar products should be 0 since $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$.

Example. We find the volume of a parallelepiped with sides corresponding to the vectors $\mathbf{u} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -1 \\ 5 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix}$. This volume is $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. We have

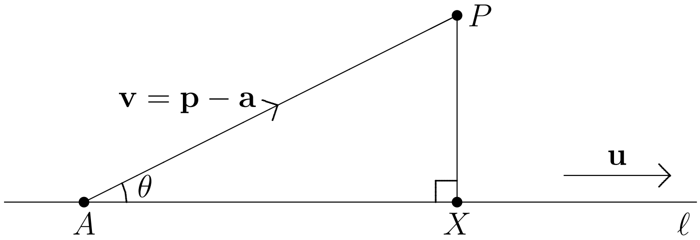
$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & -5 \\ 1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & -5 \\ 5 & 3 \end{vmatrix} \mathbf{k} = 7\mathbf{i} - 3\mathbf{j} + 22\mathbf{k}.$$

Thus $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 3(7) + 5(-3) + (-1)(22) = 21 - 15 - 22 = -16$, and so the volume is $|-16| = 16$. Note that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is a left-handed triple, since $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) < 0$.

6.8 Applications of the vector product

6.8.1 Distance from a point to a line

Let P be a point, and let ℓ be a line with vector equation $\mathbf{r} = \mathbf{a} + \lambda \mathbf{u}$ (so that $\mathbf{u} \neq \mathbf{0}$), where $\mathbf{a}$ is the position vector of the point A lying on ℓ (see diagram).



Let $\mathbf{v}$ be the vector represented by $\overrightarrow{AP}$, so that $\mathbf{v} = \mathbf{p} - \mathbf{a}$, where $\mathbf{p}$ is the position vector of P. Suppose that P is not on ℓ, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. If X is a [or rather the] point on ℓ nearest to P then angle AXP is $\frac{\pi}{2}$, and the distance from P to ℓ is:

$$|\overrightarrow{PX}| = |\overrightarrow{AP}| \sin\theta = |\mathbf{v}| \sin\theta = \frac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{u}|} = \frac{|\mathbf{v} \times \mathbf{u}|}{|\mathbf{u}|} = \frac{|(\mathbf{p} - \mathbf{a}) \times \mathbf{u}|}{|\mathbf{u}|}.$$

Note that when P is on ℓ then $\mathbf{v}$ is a scalar multiple of $\mathbf{u}$, and $|\mathbf{u} \times \mathbf{v}|/|\mathbf{u}| = 0$, which is the correct distance from P to ℓ in this case also.

You should compare this to the result obtained in Section 3.5, taking due account of the different labelling used in that section. The result that $|\mathbf{a} \times \mathbf{b}| = \sqrt{|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2}$ for all vectors $\mathbf{a}$ and $\mathbf{b}$ (proof exercise) should also prove to be useful here.

Example. Find the distance from $P = (-3, 7, 4)$ to the line ℓ with vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix}$. Here $\mathbf{a} = \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix}$, $A = (2, -2, -3)$ and $\mathbf{u} = \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix}$. So $\overrightarrow{AP}$ represents $\mathbf{v} = \mathbf{p} - \mathbf{a} = \begin{pmatrix} -3 \\ 7 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -5 \\ 9 \\ 7 \end{pmatrix}$. Now

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} -5 & 9 \\ 3 & 7 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 4 & -5 \\ 3 & 7 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 4 & -5 \\ -5 & 9 \end{vmatrix} \mathbf{k} = \begin{pmatrix} -62 \\ -43 \\ 11 \end{pmatrix}.$$

Therefore $|\mathbf{u} \times \mathbf{v}| = \sqrt{(-62)^2 + (-43)^2 + 11^2} = \sqrt{3844 + 1849 + 121} = \sqrt{5814} = 3\sqrt{646}$ and $|\mathbf{u}| = \sqrt{4^2 + (-5)^2 + 3^2} = \sqrt{50} = 5\sqrt{2}$, and thus we conclude that the distance is

$$\frac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{u}|} = \frac{3\sqrt{646}}{5\sqrt{2}} = \frac{3\sqrt{323}}{5}.$$

(Note that we have $\mathbf{v} \times \mathbf{u} = (\mathbf{p} - \mathbf{a}) \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$, and so $|\mathbf{v} \times \mathbf{u}| = |(\mathbf{p} - \mathbf{a}) \times \mathbf{u}| = |\mathbf{u} \times \mathbf{v}|$.)

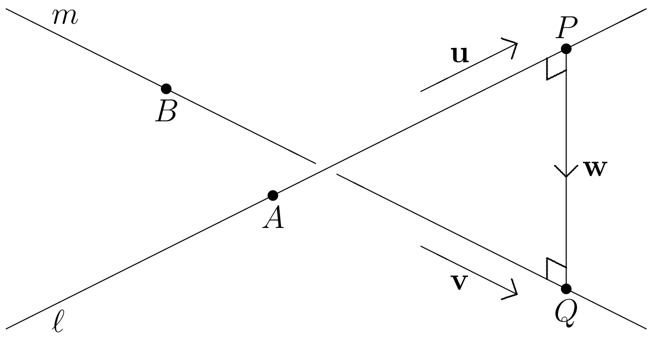
6.8.2 Distance between two lines

We derive a formula for the distance between lines ℓ and m having vector equations

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{u} \quad \text{and} \quad \mathbf{r} = \mathbf{b} + \mu\mathbf{v}$$

respectively. (By distance between ℓ and m, we mean the shortest distance from a point on ℓ to a point on m.) For the vector equations above to be valid, we require that $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$. If $\mathbf{u}$ and $\mathbf{v}$ are parallel, then this distance is the distance from P to m, where P is *any* point on ℓ.

So from now on, we assume that $\mathbf{u}$ and $\mathbf{v}$ are not parallel (and are nonzero). Let $\overrightarrow{PQ}$ be a shortest directed line segment from a point on ℓ to a point on m. This situation is shown in the diagram below (on the next page).



Then the vector $\mathbf{w}$ represented by $\overrightarrow{PQ}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, and so must be a scalar multiple of $\mathbf{u} \times \mathbf{v}$. Thus

$$\mathbf{w} = \mathbf{q} - \mathbf{p} = \alpha(\mathbf{u} \times \mathbf{v}),$$

for some scalar α, where $\mathbf{p}$ and $\mathbf{q}$ are (respectively) the position vectors of P and Q. Now P is on ℓ and so $\mathbf{p} = \mathbf{a} + \lambda\mathbf{u}$ for some λ, and Q is on m and so $\mathbf{q} = \mathbf{b} + \mu\mathbf{v}$ for some μ. Therefore, $\mathbf{q} - \mathbf{p} = \alpha(\mathbf{u} \times \mathbf{v}) = \mathbf{b} - \mathbf{a} + \mu\mathbf{v} - \lambda\mathbf{u}$, and thus:

$$\begin{aligned}
\alpha|\mathbf{u} \times \mathbf{v}|^2 &= \alpha(\mathbf{u} \times \mathbf{v})\cdot(\mathbf{u} \times \mathbf{v}) \\
&= (\mathbf{b} - \mathbf{a} + \mu\mathbf{v} - \lambda\mathbf{u})\cdot(\mathbf{u} \times \mathbf{v}) \\
&= (\mathbf{b} - \mathbf{a})\cdot(\mathbf{u} \times \mathbf{v}) + \mu(\mathbf{v}\cdot(\mathbf{u} \times \mathbf{v})) - \lambda(\mathbf{u}\cdot(\mathbf{u} \times \mathbf{v})) \\
&= (\mathbf{b} - \mathbf{a})\cdot(\mathbf{u} \times \mathbf{v}),
\end{aligned}$$

where the last step holds since $\mathbf{u}\cdot(\mathbf{u} \times \mathbf{v}) = \mathbf{v}\cdot(\mathbf{u} \times \mathbf{v}) = 0$. So we obtain

$$\alpha = \frac{(\mathbf{b} - \mathbf{a})\cdot(\mathbf{u} \times \mathbf{v})}{|\mathbf{u} \times \mathbf{v}|^2}.$$

Thus the length of $\overrightarrow{PQ}$ is

$$|\overrightarrow{PQ}| = |\mathbf{w}| = |\alpha||\mathbf{u} \times \mathbf{v}| = \frac{|(\mathbf{b} - \mathbf{a})\cdot(\mathbf{u} \times \mathbf{v})|}{|\mathbf{u} \times \mathbf{v}|},$$

and this is the distance between ℓ and m. This formula does not apply when $\mathbf{u}$ and $\mathbf{v}$ are parallel (it becomes $\frac{0}{0}$, which is not helpful).

Example. We calculate the distance between the lines ℓ and m having vector equations $\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$ and $\mathbf{r} = \mathbf{b} + \mu\mathbf{v}$ respectively, where

$$\mathbf{a} = \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}.$$

We have $\mathbf{b} - \mathbf{a} = 2\mathbf{i} - 5\mathbf{j} + \mathbf{k}$ and

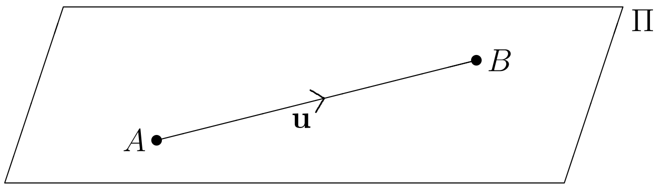
$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} \times \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{vmatrix} -3 & 1 \\ -2 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -3 \\ -2 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -3 \\ -3 & 1 \end{vmatrix} \mathbf{k} = \begin{pmatrix} -4 \\ 4 \\ -8 \end{pmatrix}.$$

Thus we get $(\mathbf{b}-\mathbf{a})\cdot(\mathbf{u}\times\mathbf{v}) = -8-20-8 = -36$ and $|\mathbf{u}\times\mathbf{v}| = \sqrt{(-4)^2+4^2+(-8)^2} = \sqrt{16+16+64} = \sqrt{96} = 4\sqrt{6}$. Therefore the distance from ℓ to m is

$$\frac{|(\mathbf{b}-\mathbf{a})\cdot(\mathbf{u}\times\mathbf{v})|}{|\mathbf{u}\times\mathbf{v}|} = \frac{|-36|}{\sqrt{96}} = \frac{36}{4\sqrt{6}} = \frac{9}{\sqrt{6}} \left[= \frac{3\sqrt{3}}{\sqrt{2}} = \frac{3\sqrt{6}}{2}\right].$$

6.8.3 Equations of planes (revisited)

Let Π be a plane, and let $\mathbf{u}$ be a nonzero vector. We say that Π is *parallel* to $\mathbf{u}$ (or $\mathbf{u}$ is *parallel* to Π) if there are points A and B on Π such that $\overrightarrow{AB}$ represents $\mathbf{u}$.



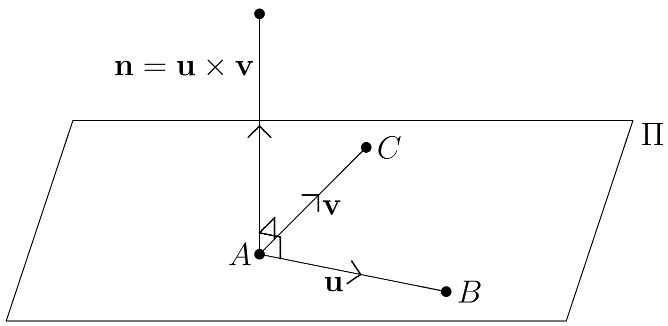
Two planes Π and Π' are *parallel* if every nonzero vector parallel to one is parallel to the other. Now suppose that $\mathbf{u}$ and $\mathbf{v}$ are nonzero non-parallel vectors. Then $\mathbf{n} := \mathbf{u}\times\mathbf{v}$ is a nonzero vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. Thus if Π is a plane through the point P and parallel to both $\mathbf{u}$ and $\mathbf{v}$ then $\mathbf{n}[\neq \mathbf{0}]$ is orthogonal to Π and a vector equation for Π is

$$\mathbf{r}\cdot\mathbf{n} = \mathbf{p}\cdot\mathbf{n},$$

where $\mathbf{p}$ is the position vector of P. (In the diagram below, P can be taken to correspond to A.)

We are also interested in the equation of a plane determined by three points which do not all lie on the same line. So let A, B and C be three points which do not all lie on the same line, let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be their position vectors, and let Π be the unique plane containing A, B and C. One possible vector equation of Π is $\mathbf{r} = \mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})+\mu(\mathbf{c}-\mathbf{a})$, and this was given in Section 3.3. But we really want an equation of the form $\mathbf{r}\cdot\mathbf{n} = \mathbf{p}\cdot\mathbf{n}$, where $\mathbf{p}$ is the position vector of a point in Π and $\mathbf{n}$ is orthogonal to Π. So we can take $\mathbf{p} = \mathbf{a}$. But how can we determine a suitable normal vector $\mathbf{n}$?

We note that Π is parallel to the vectors $\mathbf{u} := \mathbf{b}-\mathbf{a}$ and $\mathbf{v} := \mathbf{c}-\mathbf{a}$ represented by $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively. Then $\mathbf{u}$ and $\mathbf{v}$ are nonzero non-parallel vectors. Therefore $\mathbf{u}\times\mathbf{v}$ is a nonzero (proof exercise) vector perpendicular to $\mathbf{u}$ and $\mathbf{v}$, and thus orthogonal to Π, and so we can take $\mathbf{n} = \mathbf{u}\times\mathbf{v}$.



Finally, we find the plane determined by two points and a vector. So let A and B be points having position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. We now examine the plane Π containing points A and B and parallel to $\mathbf{v}$, where $\mathbf{u} := \mathbf{b} - \mathbf{a}$ and $\mathbf{v}$ are not collinear (which in particular means that $\mathbf{v} \neq \mathbf{0}$ and $A \neq B$). Then Π has vector equation $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$ where $\mathbf{n} = \mathbf{u} \times \mathbf{v} = (\mathbf{b} - \mathbf{a}) \times \mathbf{v}$. (The diagram is relevant to this case too.)

Example 1. We determine a Cartesian equation for the plane Π through the point $A = (3, 4, 1)$ and parallel to $\mathbf{u} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$. We have

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \mathbf{k} = 10\mathbf{i} - 5\mathbf{j} - 5\mathbf{k} = \begin{pmatrix} 10 \\ -5 \\ -5 \end{pmatrix}.$$

A vector equation for Π is

$$\mathbf{r} \cdot \begin{pmatrix} 10 \\ -5 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ -5 \\ -5 \end{pmatrix},$$

and so a Cartesian equation is $10x - 5y - 5z = 5$, which is equivalent to $2x - y - z = 1$ (after dividing through by 5).

Example 2. We determine a Cartesian equation for the plane Π through the points $A = (1, 2, 4)$, $B = (2, 4, 1)$ and $C = (4, 1, 2)$. Then Π is parallel to:

$$\mathbf{u} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \text{ and } \mathbf{v} = \mathbf{c} - \mathbf{a} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix},$$

and so Π is orthogonal to $\mathbf{n} = \mathbf{u} \times \mathbf{v} = -7\mathbf{i} - 7\mathbf{j} - 7\mathbf{k}$. Since A is also on Π a Cartesian equation for Π is $-7x - 7y - 7z = \mathbf{a}\cdot\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -7 \\ -7 \\ -7 \end{pmatrix} = -49$, or equivalently $x + y + z = 7$. Since we can take $\mathbf{n}$ to be a nonzero scalar multiple of $\mathbf{u} \times \mathbf{v}$, we could have taken $\mathbf{n} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, which would have led directly to the Cartesian equation $x + y + z = 7$. (As a check, we can verify that A, B and C lie on the plane defined by this equation.)

6.9 Is the cross product commutative or associative?

We have already indicated that it is neither. For example, $\mathbf{i} \times \mathbf{j} = \mathbf{k} \neq -\mathbf{k} = \mathbf{j} \times \mathbf{i}$ and $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = -\mathbf{j} \neq \mathbf{0} = (\mathbf{i} \times \mathbf{i}) \times \mathbf{j}$. It is important to give *explicit* examples when the equalities fail to hold (or to somehow establish by stealth that such examples exist). In particular, we should ***not*** use the anti-commutative law (Theorem 6.2) to prove that the vector product is not commutative. This is because:

(i) It could be the case that $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ for all $\mathbf{u}$ and $\mathbf{v}$, in which case the vector product would be both commutative and anti-commutative. (It may seem obvious that the vector product is not always zero; the point is that this property must still be explicitly *checked* [once].)

(ii) Even if $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$, it could still be the case that $\mathbf{u} \times \mathbf{v}$ is equal to a vector $\mathbf{w}$ having the properties that $\mathbf{w} = -\mathbf{w}$ and $\mathbf{w} \neq \mathbf{0}$. (There are mathematical systems in which this can happen. However, $\mathbb{R}^3$ is not one of them, see Lemma 6.10 below.)

We shall classify all pairs $\mathbf{u}, \mathbf{v}$ of vectors such that $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$ and all triples $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of vectors such that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

Lemma 6.10. Let $\mathbf{w}$ be a vector (in $\mathbb{R}^3$, or even $\mathbb{R}^n$) such that $\mathbf{w} = -\mathbf{w}$. Then $\mathbf{w} = \mathbf{0}$.

Proof. Adding $\mathbf{w}$ to both sides of $\mathbf{w} = -\mathbf{w}$ gives $2\mathbf{w} = \mathbf{0}$, and then multiplying both sides by $\frac{1}{2}$ gives $\mathbf{w} = \mathbf{0}$, as required. $\square$

Theorem 6.11. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors. Then $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are collinear, which is if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

Proof. If $\mathbf{u}$ and $\mathbf{v}$ are collinear then $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u} = \mathbf{0}$. If $\mathbf{u}$ and $\mathbf{v}$ are not collinear $\mathbf{u} \neq \mathbf{0}$, $\mathbf{v} \neq \mathbf{0}$, and the angle θ between $\mathbf{u}$ and $\mathbf{v}$ satisfies $0 < \theta < \pi$, and so $\sin\theta > 0$. Thus, from the definition of $\mathbf{u} \times \mathbf{v}$, we get $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta > 0$, whence $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$. Then the previous lemma (Lemma 6.10) gives us that $\mathbf{u} \times \mathbf{v} \neq -(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \times \mathbf{u}$. $\square$

6.9.1 The triple vector product(s)

Definition. The *triple vector products* of the ordered triple of vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are defined to be $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ and $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

Theorem 6.12. For all vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ we have $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ and $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u}$

Proof. A somewhat tedious calculation using coördinates can prove this result, and this is left as an exercise for the reader. $\square$

Some aspects of this result can be proved geometrically. We let $\mathbf{p} = \mathbf{v} \times \mathbf{w}$ and $\mathbf{q} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$, and suppose that $\mathbf{v}$ and $\mathbf{w}$ are not collinear. (It is annoying that the case when $\mathbf{v}$, $\mathbf{w}$ are collinear must be treated separately. But the first equality is fairly easy to prove in this case.) Since $\mathbf{p} = \mathbf{v} \times \mathbf{w}$ is orthogonal to $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{q} = \mathbf{u} \times \mathbf{p}$, we see that the point with position vector $\mathbf{q}$ must be in the plane determined by O, $\mathbf{v}$ and $\mathbf{w}$. That is $\mathbf{q} = \alpha\mathbf{v} + \beta\mathbf{w}$ for some scalars α and β. The requirement that $\mathbf{q}$ be orthogonal to $\mathbf{u}$ determines the ratio between α and β.

Theorem 6.13. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Then $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ if and only $(\mathbf{u}\cdot\mathbf{v})\mathbf{w} = (\mathbf{v}\cdot\mathbf{w})\mathbf{u}$. The latter condition holds if and only if one (or both) of the following hold:

1. $\mathbf{u}$ and $\mathbf{w}$ are collinear; or
2. $\mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{w}$.

Proof. We take the formulae for $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ and $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ given in Theorem 6.12. Thus $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ if and only if

$$(\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w} = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u}. \tag{6.3}$$

We now perform the reversible operations of subtracting $(\mathbf{u}\cdot\mathbf{w})\mathbf{v}$ from both sides of (6.3) followed by negating both sides, to get that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ if and only if $(\mathbf{u}\cdot\mathbf{v})\mathbf{w} = (\mathbf{v}\cdot\mathbf{w})\mathbf{u}$. If $\mathbf{u}$ and $\mathbf{w}$ are collinear, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$ or $\mathbf{u} = \lambda\mathbf{w}$ for some λ. In the first two cases we get $(\mathbf{u}\cdot\mathbf{v})\mathbf{w} = (\mathbf{v}\cdot\mathbf{w})\mathbf{u} = 0$, and in the last case we get $(\mathbf{u}\cdot\mathbf{v})\mathbf{w} = (\mathbf{v}\cdot\mathbf{w})\mathbf{u} = (\lambda(\mathbf{u}\cdot\mathbf{v}))\mathbf{u}$. If $\mathbf{u}$ and $\mathbf{w}$ are not collinear, then $(\mathbf{u}\cdot\mathbf{v})\mathbf{w} = (\mathbf{v}\cdot\mathbf{w})\mathbf{u}$ if and only if $-(\mathbf{v}\cdot\mathbf{w})\mathbf{u} + (\mathbf{u}\cdot\mathbf{v})\mathbf{w} = \mathbf{0}$, which happens if and only if $\mathbf{v}\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{v} = 0$ by the definition of collinear, see Equation 6.1 of Section 6.2. But $\mathbf{v}\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{v} = 0$ if and only if $\mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{w}$. $\square$