Chapter 5

Gaußian Elimination and Echelon Form

A *linear equation* in variables $x_1, x_2, \ldots, x_n$ is of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = d,$$

where $a_1, a_2, \ldots, a_n, d$ are scalars. The x_1-term a_1x_1 is the first term, The x_2-term a_2x_2 is the second term, and in general the x_i-term a_ix_i is the i^{th} term. In the case when $n = 3$, we generally use a, b, c, x, y, z instead of $a_1, a_2, a_3, x_1, x_2, x_3$; thus $a_1x_1 + a_2x_2 + a_3x_3 = d$ becomes $ax + by + cz = d$.

A linear equation $a_1x_1+a_2x_2+\cdots+a_nx_n = d$ is *degenerate* if $a_1 = a_2 = \cdots = a_n = 0$; otherwise it is *non-degenerate*. The equation can be degenerate even if $d \neq 0$.

5.1 Echelon form (Geometry I definition)

Before defining echelon form we make some comments. Firstly, echelon form is properly a concept for matrices; we give an equivalent definition for systems of linear equations. Secondly, there are 3 different definitions of echelon form, of varying strengths, and it is possible that you may have encountered a different one previously. We use the weakest of the definitions in this course. **For this course, it is important that you understand and use the definition of echelon form given below.**

Definition 5.1. A system of linear equations in $x_1, x_2, \ldots, x_n$ is in *echelon form* if every non-degenerate equation begins with strictly fewer zero terms than each equation below it and any degenerate equation occurs after (below) the non-degenerate equations.

Note that any system of linear equations in echelon form has at most n non-degenerate equations (if any), but can have arbitrarily many degenerate equations (if any). A system with no non-degenerate equations (or no equations whatsoever) is automatically in echelon form. The right-hand sides of a system of linear equations play *no* rôle in determining whether that system is in echelon form. We *do not* insist that the first nonzero term of each non-degenerate equation be x_i for some i. (This extra condition is currently required in the MTH5112: Linear Algebra I definition of echelon form.)

Example. As linear equations in the variables x, y, z, we have that:

- $x+2y=8$ begins with 0 zero terms (the zero z-term does not begin the equation);
- $3y-4z=-5$ begins with 1 zero term;
- $-5z=2$ begins with 2 zero terms;
- $0=0$ and $0=5$ both begin with 3 zero terms (the right-hand sides are irrelevant).

Thus the system of equations

$$\left.\begin{aligned} x+2y \qquad &= 8 \\ 3y-4z &= -5 \\ -5z &= 2 \\ 0 &= 0 \\ 0 &= 5 \end{aligned}\right\}$$

is in echelon form.

Also, the following systems of equations are in echelon form:

$$\left.\begin{aligned} x+2y \qquad &= 3 \\ 3y-4z &= -5 \\ -5z &= 4 \end{aligned}\right\} \qquad \left.\begin{aligned} x+2y \qquad &= 3 \\ -5z &= 4 \\ 0 &= 0 \end{aligned}\right\} \qquad \left.\begin{aligned} x+2y \qquad &= 3 \\ -5z &= 4 \\ 0 &= -5 \end{aligned}\right\},$$

and these are in echelon form too:

$$\left.\begin{aligned} 3y-4z &= -5 \\ -5z &= 7 \end{aligned}\right\} \qquad \left.\begin{aligned} 3y-4z &= -5 \\ 0 &= 7 \end{aligned}\right\} \qquad \left.\begin{aligned} 3y-4z &= -5 \\ 0 &= 0 \end{aligned}\right\} \qquad 3y-4z=-5\ \Big\}.$$

The following systems of equations are *not* in echelon form.

$$\left.\begin{aligned} x+2y \qquad &= 7 \\ -3z &= 4 \\ 2y-5z &= -3 \end{aligned}\right\} \qquad \left.\begin{aligned} x+4y-z &= -1 \\ 0 &= 4 \\ 3y-4z &= -5 \end{aligned}\right\} \qquad \left.\begin{aligned} x+3y \qquad &= 1 \\ y+4z &= -2 \\ y-5z &= 3 \end{aligned}\right\}.$$

In the last case, the last two equations commence with the same number (one) of zero terms but are non-degenerate.

5.2 Gaußian elimination

We now describe the process of *Gaußian elimination*, which is used to bring a system of linear equations (in $x_1, x_2, \ldots, x_n$ or x, y, z, etc.) into echelon form. **You must use exactly the algorithm described here, even though other ways of reducing to (a possibly different) echelon form may be mathematically valid.**[1] In particular, our algorithm does not use the M-operations of Section 4.3; these are required in general for reducing to the stronger versions of echelon form.

[1] The course MTH5112: Linear Algebra I uses a strictly stronger definition of echelon form than we do, and consequently uses a slightly different version of Gaußian elimination than we do in MTH4103: Geometry I. This modified version does require the M-operations of Section 4.3. You **must not** use the Linear Algebra I version of Gaußian elimination in this course.

Step 1. If the system is in echelon form then stop. (There is nothing to do.)

If each equation has zero x_1-term then go to Step 2. (There are no x_1-terms to eliminate.)

If the first equation has a zero x_1-term then interchange it with the first equation that has nonzero x_1-term. (So now the first equation has nonzero x_1-term.)

Add appropriate multiples of the first equation to the others to eliminate their x_1-terms.

Step 2. (At this point, all equations except perhaps the first should have zero x_1-term.)

If the system is in echelon form then stop.

If each equation with zero x_1-term also has zero x_2-term then go to Step 3.

If the first equation with zero x_1-term has a zero x_2-term then interchange it with the first equation that has zero x_1-term and nonzero x_2-term. (So now the first equation with zero x_1-term has nonzero x_2-term.)

Add appropriate multiples of the first equation with zero x_1-term to the equations below it to eliminate their x_2-terms.

Step m for $3 \leqslant m \leqslant n$. (At this point, all equations except perhaps the first up to $m-1$ should have zero x_1-, x_2-, $\ldots$, x_{m-1}-terms. Each of these first $\leqslant m-1$ equations should begin with strictly more zero terms than their predecessors.)

If the system is in echelon form then stop.

If each equation with zero x_1-, x_2-, $\ldots$, x_{m-1}-terms also has zero x_m-term then go to Step $m+1$. (This situation should only arise in the case when $m < n$; if $m = n$ this case should only 'arise' if you are already in echelon form [in which case you should have stopped already].)

If the first equation with zero x_1-, x_2-, $\ldots$, x_{m-1}-terms also has zero x_m-term, then interchange it with the first equation having zero x_1-, x_2-, $\ldots$, x_{m-1}-terms and nonzero x_m-term.

Add appropriate multiples of the first equation with zero x_1-, x_2-, $\ldots$, x_{m-1}-terms to the equations below it to eliminate their x_m-terms.

If $m = n$ then the system of equations should now be in echelon form, so you can stop.

5.2.1 Notes on Gaußian elimination

1. If a degenerate equation of the form $0 = 0$ is created at any stage, it can be discarded, and the Gaußian elimination continued without it.
2. If a degenerate equation of the form $0 = d$, with $d \neq 0$, is created at any stage then the system of equations has no solutions, and the Gaußian elimination can be stopped.
3. The operations required to bring a system of equations into echelon form are independent of the right-hand sides of the equations. (Compare Examples 2 and $2'$ in Section 4.2.)

Example. We perform Gaußian elimination on the following system of equations.

$$\left.\begin{aligned} x + 2y + z &= 2 \\ x + 2y + 3z &= -8 \\ 3x + 5y + 2z &= 6 \\ -2x - 2y + z &= 0 \end{aligned}\right\}. \tag{5.1}$$

Step 1. To eliminate the x-term in all equations but the first, we add -1 times the first equation to the second, -3 times the first equation to the third, and 2 times the first equation to the fourth. This gives:

$$\left.\begin{aligned} x + 2y + z &= 2 \\ 2z &= -10 \\ - y - z &= 0 \\ 2y + 3z &= 4 \end{aligned}\right\}.$$

Step 2. The first equation having zero x-term (the second) also has zero y-term. The first equation having zero x-term and nonzero y-term is the third, so interchange the second and third equations, to get:

$$\left.\begin{aligned} x + 2y + z &= 2 \\ - y - z &= 0 \\ 2z &= -10 \\ 2y + 3z &= 4 \end{aligned}\right\}.$$

Now that the second equation has zero x-term and nonzero y-term, use this to eliminate y from the equations below it. To do this we need only add twice the second equation to the fourth, to obtain:

$$\left.\begin{aligned} x + 2y + z &= 2 \\ - y - z &= 0 \\ 2z &= -10 \\ z &= 4 \end{aligned}\right\}.$$

Step 3. We are still not in echelon form, and the first equation with zero x-term and y-term has nonzero z-term. (This is the third equation.) We use the third equation to eliminate z from all equations after the third (by adding $-\frac{1}{2}$ times the third equation to the fourth). This gives:

$$\left.\begin{aligned} x + 2y + z &= 2 \\ - y - z &= 0 \\ 2z &= -10 \\ 0 &= 9 \end{aligned}\right\}.$$

This system of equations is now in echelon form, but has no solutions (as $0 = 9$ has no solutions). Hence the original system (5.1) has no solutions. Geometrically, this reflects the fact that four planes of $\mathbb{R}^3$ generally have empty intersection.

5.3 Solving systems of equations in echelon form

We now show how to find all solutions to a system of linear equations in echelon form. If the system contains an equation an equation of the form $0 = d$ with $d \neq 0$, then the system has no solutions, and we have nothing more to do. If the system has any equations of the form $0 = 0$, then throw these out, since they contribute no restriction on the solution set whatsoever. Thus we may now only consider systems of linear equations in echelon form which have no degenerate equations.

Definition. In a non-degenerate linear equation (written in standard form), the first variable, reading left to right, in a nonzero term is called the *leading variable* of the equation.

Example. In these examples, we assume that we have three variables x, y, z, in that order. In the first three cases, the linear equation is in standard form; in the latter two cases it is not.

- The leading variable of $-x + y = 3$ is x.
- The leading variable of $3y - 2z = 7$ is y.
- The leading variable of $-4z = 8$ is z.
- The standard form of $z - y + 2x - y = 4$ is $2x - 2y + z = 4$, so the leading variable is x.
- The equation $x + (-1)x - 3 = 4$ is degenerate (it is equivalent to $-3 = 4$, or $0 = 7$), and so has no leading variable defined.

5.3.1 Back substitution

Back substitution is an algorithm to determine *all* the solutions to a system of non-degenerate equations in echelon form. It proceeds as follows.

Step 1. Variables which are not leading variables of any of the equations in the system can take arbitrary (real) values. Assign a symbolic value to each such non-leading variable.[2]

Step 2. Given symbolic values for the non-leading variables, solve for the leading variables, starting from the bottom and working up.

[2] In general, we shall want to apply this procedure when the equations are defined over fields F other than $\mathbb{R}$. In this case, the non-leading variables should take arbitrary values in F. You should see the concept of *field* defined in MTH4104: Introduction to Algebra. (MTH4104 is compulsory for about half of you this year, and strongly recommended for the rest of you next year.) Examples of fields are $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ (but not $\mathbb{N}$ or $\mathbb{Z}$). You should have met all these sets in MTH4110: Mathematical Structures, and gained some extra familiarity with $\mathbb{C}$ in MTH4101: Calculus II.

Example 1. We apply back substitution to the following system of equations, which is in echelon form:

$$\left.\begin{aligned} x + 3y - z &= 7 \\ z &= 0 \end{aligned}\right\}.$$

There is just one non-leading variable, namely y. Thus y can take any real value, say $y = t$. We now solve for the leading variables x and z, starting with z.

The last equation gives $z = 0$. Therefore, the first equation gives $x + 3t + 0 = 7$, and so $x = 7 - 3t$. Thus, the solutions of the system are $x = 7 - 3t$, $y = t$, $z = 0$, where t can be any real number.

We remark that the intersection of the planes defined by these equations is thus $\{\,(7-3t, t, 0) : t \in \mathbb{R}\,\}$, which is the set of points on the line having parametric equations:

$$\left.\begin{aligned} x &= 7 - 3\lambda \\ y &= \lambda \\ z &= 0 \end{aligned}\right\},$$

and Cartesian equations $\frac{x-7}{-3} = y$, $z = 0$.

Example 2. We apply back substitution to the following system of equations, which is in echelon form:

$$\left.\begin{aligned} x + 3y + 5z &= 9 \\ 2y + 4z &= 6 \\ 3z &= 3 \end{aligned}\right\}.$$

All the variables are leading in one of the equations. So we start immediately on Step 2. The (third) equation $3z = 3$ gives $z = 1$. Then the second equation becomes $2y + 4 = 6$, whence we get that $y = 1$. Then the first equation becomes $x + 3 + 5 = 9$, and so $x = 1$. Therefore the only solution of this system of equations is $x = y = z = 1$.

Example 3. We apply back substitution to the following system of just one equation, which is in echelon form:

$$2x - y + 3z = 5 \;\}.$$

The non-leading variables y and z can take arbitrary real values, say $y = s$ and $z = t$. We now solve for the only leading variable, x, using the only equation of the system. We have $2x - s + 3t = 5$, so $2x = 5 + s - 3t$, and hence $x = \frac{5}{2} + \frac{s}{2} - \frac{3t}{2}$. Thus the solutions of the system are $x = \frac{5}{2} + \frac{s}{2} - \frac{3t}{2}$, $y = s$, $z = t$, where s and t can be any real numbers.

We remark that is follows that $\{\,(\frac{5}{2} + \frac{s}{2} - \frac{3t}{2}, s, t) : s, t \in \mathbb{R}\,\}$ is the set of points on the plane defined by $2x - y + 3z = 5$.

5.4 Summary

To solve a system of linear equations, first use Gaußian elimination to bring the system into echelon form, and then use back substitution to the system in echelon form, remembering to deal appropriately with degenerate equations along the way. You should review your notes, starting at the beginning of Chapter 4, to see many examples of this.

5.5 Intersection of a line and a plane

Consider a line ℓ defined by the parametric equations:

$$\left.\begin{aligned} x &= p_1 + u_1\lambda \\ y &= p_2 + u_2\lambda \\ z &= p_3 + u_3\lambda \end{aligned}\right\},$$

and a plane Π defined by

$$ax + by + cz = d.$$

We find the intersection of ℓ and Π by determining the λ for which

$$a(p_1 + u_1\lambda) + b(p_2 + u_2\lambda) + c(p_3 + u_3\lambda) = d. \tag{5.2}$$

This gives one linear equation in one unknown λ. The solution is unique, except in the degenerate case when $au_1 + bu_2 + cu_3 = 0$, in which case ℓ is parallel to Π. In this case, there are 0 or infinitely many solutions, depending on whether ℓ is in Π or not (which is if and only if $ap_1 + bp_2 + cp_3 = d$ or not). Note that rearranging Equation 5.2 to standard form for a linear equation in λ yields

$$(au_1 + bu_2 + cu_3)\lambda = d - (ap_1 + bp_2 + cp_3). \tag{5.3}$$

So if Π and ℓ have vector equations $\mathbf{r}\cdot\mathbf{n} = d$ and $\mathbf{r} = \mathbf{p} + \lambda\mathbf{u}$ respectively, we find that

$$(\mathbf{n}\cdot\mathbf{u})\lambda = d - \mathbf{n}\cdot\mathbf{p}. \tag{5.4}$$

(You should be able to work out what $\mathbf{n}$, $\mathbf{p}$ and $\mathbf{u}$ are here.) Thus if $\mathbf{n}\cdot\mathbf{u} \neq 0$ we get the unique solution $\lambda = (d - \mathbf{n}\cdot\mathbf{p})/(\mathbf{n}\cdot\mathbf{u})$.

Example. For example, let ℓ be the line with parametric equations:

$$\left.\begin{aligned} x &= 1 + 2\lambda \\ y &= 2 + 3\lambda \\ z &= -1 - 4\lambda \end{aligned}\right\},$$

and let Π be the plane defined by $x - y + 2z = 3$. To determine the intersection $\Pi \cap \ell$ of Π and ℓ we solve

$$(1 + 2\lambda) - (2 + 3\lambda) + 2(-1 - 4\lambda) = 3.$$

This gives $-3 - 9\lambda = 3$, or $-9\lambda = 6$, with unique solution $\lambda = -\frac{2}{3}$. (If we wish to use the formula below Equation 5.4 to calculate λ we note that $d = 3$, $\mathbf{n}\cdot\mathbf{p} = -3$ and $\mathbf{n}\cdot\mathbf{u} = -9$.) Thus ℓ and Π intersect in the single point (x_0, y_0, z_0), with

$$\begin{aligned} x_0 &= 1 + 2(-\tfrac{2}{3}) = -\tfrac{1}{3}, \\ y_0 &= 2 + 3(-\tfrac{2}{3}) = 0, \\ z_0 &= -1 - 4(-\tfrac{2}{3}) = \tfrac{5}{3}. \end{aligned}$$

As a set of points, the intersection $\ell \cap \Pi$ of ℓ and Π is $\{(-\frac{1}{3}, 0, \frac{5}{3})\}$.

Notation. Let A and B be sets. Then the *intersection* of A and B is denoted $A \cap B$, and is the set of elements that are in both A and B. The *union* of A and B is denoted $A \cup B$, and is the set of elements that are in A or B (or both). As a mnemonic, the symbol $\cap$ resembles a lower case N, which is the second letter of intersection, and the symbol $\cup$ resembles a lower case U, the first letter of union.

The following properties hold for $\cap$ and $\cup$, for *all* sets A, B, C. The last pair of properties only makes sense in the presence of a 'universal' set $\mathscr{E}$.

- $A \cap B = B \cap A$ and $A \cup B = B \cup A$.
- $A \cap (B \cap C) = (A \cap B) \cap C$ and $A \cup (B \cup C) = (A \cup B) \cup C$.
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
- $A \cap A = A \cup A = A$.
- $A \cap \varnothing = \varnothing$ and $A \cup \varnothing = A$.
- $A \cap \mathscr{E} = A$ and $A \cup \mathscr{E} = \mathscr{E}$.

5.6 Intersection of two lines

Consider lines ℓ and m defined (respectively) by the parametric equations:

$$\left.\begin{aligned} x &= p_1 + u_1\lambda \\ y &= p_2 + u_2\lambda \\ z &= p_3 + u_3\lambda \end{aligned}\right\} \qquad \text{and} \qquad \left.\begin{aligned} x &= q_1 + v_1\mu \\ y &= q_2 + v_2\mu \\ z &= q_3 + v_3\mu \end{aligned}\right\}.$$

We find the intersection of ℓ and m by determining the λ and μ for which:

$$\begin{aligned} p_1 + u_1\lambda &= q_1 + v_1\mu, \\ p_2 + u_2\lambda &= q_2 + v_2\mu, \\ p_3 + u_3\lambda &= q_3 + v_3\mu. \end{aligned}$$

This is equivalent to solving the following system of linear equations:

$$\left.\begin{aligned} u_1\lambda - v_1\mu &= q_1 - p_1 \\ u_2\lambda - v_2\mu &= q_2 - p_2 \\ u_3\lambda - v_3\mu &= q_3 - p_3 \end{aligned}\right\}.$$

Note that, in general, we expect this system of equations to have no solution.

Example. In this example, the lines ℓ and m are defined (respectively) by the parametric equations:

$$\left.\begin{aligned} x &= 1 + \lambda \\ y &= 2 + 3\lambda \\ z &= 1 - 4\lambda \end{aligned}\right\} \qquad \text{and} \qquad \left.\begin{aligned} x &= 2 + 3\mu \\ y &= 1 - \mu \\ z &= 3 + 2\mu \end{aligned}\right\}.$$

Thus we must solve (for λ and μ) the following equations:

$$\left.\begin{aligned} 1+\lambda \;\; &= 2+3\mu, \\ 2+3\lambda &= 1-\mu, \\ 1-4\lambda &= 3+2\mu. \end{aligned}\right\}$$

This is equivalent to the following system of linear equations:

$$\left.\begin{array}{rcr} \lambda - 3\mu &=& 1 \\ 3\lambda + \;\; \mu &=& -1 \\ -4\lambda - 2\mu &=& 2 \end{array}\right\}.$$

We now apply Gaußian elimination. Adding -3 times the first equation to the second and 4 times the first equation to the third gives:

$$\left.\begin{array}{rcr} \lambda - \;\; 3\mu &=& 1 \\ 10\mu &=& -4 \\ -\; 14\mu &=& 6 \end{array}\right\}.$$

We now need to eliminate μ in the third equation. To do this add $\frac{7}{5}[= -(\frac{-14}{10})]$ times the second equation to the third, to get:

$$\left.\begin{array}{rcr} \lambda - \;\; 3\mu &=& 1 \\ 10\mu &=& -4 \\ 0 &=& \frac{2}{5} \end{array}\right\}.$$

There is no solution to this system (because of the equation $0 = \frac{2}{5}$). Thus we conclude that the lines ℓ and m do not meet. As a set of points, the intersection of ℓ and m is $\varnothing$, the empty set.

Here, the lines ℓ and m are *skew*, that is, they do not meet in any point, but they are not parallel either (since their direction vectors are not scalar multiples of each other).