Chapter 4

Intersections of Planes and Systems of Linear Equations

Using coördinates, a plane Π is defined by an equation:

$$ax + by + cz = d,$$

where a, b, c, d are real numbers, and at least one of a, b, c is nonzero. The set of points on Π consists precisely of the points (p, q, r) with $ap + bq + cr = d$. In set-theoretic notation, this set is:

$$\{\,(p, q, r) : p, q, r \in \mathbb{R} \mid ap + bq + cr = d\,\}.$$

Suppose we wish to determine the intersection (as a set of points) of k given planes $\Pi_1, \Pi_2, \ldots, \Pi_k$ given by the respective equations:

$$\left.\begin{array}{c} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ \vdots \\ a_kx + b_ky + c_kz = d_k \end{array}\right\}. \tag{4.1}$$

Now a point (a, b, c) is in this intersection precisely when it is on each of the planes $\Pi_1, \Pi_2, \ldots, \Pi_k$, which is the case precisely when $x = p$, $y = q$, $z = r$ is a solution to each of the equations $a_ix + b_iy + c_iz$ of (4.1), where $1 \leqslant i \leqslant k$. Thus, to determine the intersection, we need to determine the solutions to the system of k linear equations (4.1) in 3 unknowns. The technique we shall use to do this, called *Gaußian elimination* or *reduction to echelon form* can be applied to determine the solutions to a system of k linear equations in any number of unknowns.

Note. Gaußian elimination is named after Carl Friedrich Gauß (1777–1855). The symbol ß is a special German symbol called *Eszett* or *scharfes S*, and is pronounced like the English word-initial S, and is often rendered into English as ss; thus Gauß is often written as Gauss in English.

4.1 Possible intersections of k planes in $\mathbb{R}^3$

For $k = 0, 1, 2, 3, 4, \ldots$ we detail below various configurations of k planes in $\mathbb{R}^3$, and what their intersections are. (The cases $k = 0, 1$ did not appear in lectures.)

- The intersection of 0 planes of $\mathbb{R}^3$ is the whole of $\mathbb{R}^3$. (See below for why.)
- The intersection of 1 plane(s) Π_1 of $\mathbb{R}^3$ is simply Π_1.
- The intersection of 2 planes Π_1, Π_2 of $\mathbb{R}^3$ is usually a line. The only exceptions occur when Π_1 and Π_2 are parallel. In such a case, if $\Pi_1 \neq \Pi_2$, then Π_1 and Π_2 intersect nowhere, whereas if $\Pi_1 = \Pi_2$, then Π_1 and Π_2 intersect in the plane Π_1. Example 3 below is a case when Π_1 and Π_2 are parallel but not equal.
- In general, 3 planes Π_1, Π_2, Π_3 intersect at precisely one point (Example 1 below is like this). Exceptional situations arise when two (or all) of the planes are parallel. Assuming that no two of Π_1, Π_2, Π_3 are parallel, exceptional situations arise only when the intersections of Π_1 with Π_2, Π_2 with Π_3 and Π_3 with Π_1 are parallel lines. These lines either coincide, in which case Π_1, Π_2, Π_3 intersect in this line (Example 2 below is like this), or the three lines are distinct, in which case Π_1, Π_2, Π_3 have empty intersection (Example $2'$ below is like this).
- In general, 4 or more planes intersect at no points whatsoever. Another way of saying this is that their intersection is $\varnothing$, the empty set. Non-empty intersections are possible in exceptional cases.

4.1.1 Empty intersections, unions, sums and products

This was not done in lectures. Empty products and so on are somewhat subtle, and cause a lot of confusion and stress. Take the following as definitions.

- If I intersect 0 sets, each of which is presumed to belong to some "universal" set, then their intersection is that "universal" set. In the case above, the "universal" set was $\mathbb{R}^3$. A *"universal" set* is a set that contains (as elements) all the entities one wishes to consider in a given situation. If no "universal" set is understood (or exists) in the context in which you happen to be working, then the intersection of 0 sets is undefined. Taking the complement of a set is only defined when a "universal" set is around.
- The union of 0 sets is the empty set $\varnothing$. (There is no need to assume the existence a "universal" set here.)
- The sum of 0 real (or rational or complex) numbers is 0, and the sum of 0 vectors is $\mathbf{0}$. [In general, the sum of 0 things is the additive identity of the object these things are taken to belong to, when such a thing exists and is unique.]
- The product of 0 real (or rational or complex) numbers is 1. [In general, the product of 0 things is the multiplicative identity of the object these things are taken to belong to, when such a thing exists and is unique.]

4.2 Some examples

Before we formalise the notions of *linear equation*, *Gaußian elimination*, *echelon form* and *back substitution* in the next chapter, we give some examples of solving systems of linear equations using these methods.

Example 1. We determine all solutions to the system of equations:

$$\left.\begin{array}{r} x + y + z = 1 \\ -2x + 2y + z = -1 \\ 3x + y + 5z = 7 \end{array}\right\}. \tag{4.2}$$

We use the first equation to eliminate x in the second and third equations. We do this by adding twice the first equation to the second, and -3 times the first equation to the third, to get:

$$\left.\begin{array}{r} x + y + z = 1 \\ 4y + 3z = 1 \\ -2y + 2z = 4 \end{array}\right\}.$$

We now use the second equation to eliminate y in third by adding $\frac{1}{2}$ times the second equation to the third, which gives:

$$\left.\begin{array}{r} x + y + z = 1 \\ 4y + 3z = 1 \\ \frac{7}{2}z = \frac{9}{2} \end{array}\right\}.$$

We have now reduced the system to something called *echelon form*, and this is easy to solve by a process known as *back substitution*. The third equation gives $z = \frac{9/2}{7/2} = \frac{9}{7}$. Then the second equation gives $4y + 3(\frac{9}{7}) = 1$, and so $4y = -\frac{20}{7}$, whence $y = -\frac{5}{7}$. Then the first equation gives $x - \frac{5}{7} + \frac{9}{7} = 1$, whence $x = \frac{3}{7}$.

We conclude that the only solution to the system of equations (4.2) is $x = \frac{3}{7}$, $y = -\frac{5}{7}$, $z = \frac{9}{7}$. Thus the three planes defined by the equations of (4.2) intersect in the single point $(\frac{3}{7}, -\frac{5}{7}, \frac{9}{7})$. Recall that, in general, three planes intersect in precisely one point.

It is always good practice to check that any solution you get satisfies the original equations. You will probably pick up most mistakes this way. If your 'solution' does not satisfy the original equations then you have certainly made a mistake. If the original equations are satisfied, then you have possibly made a mistake and got lucky, and you could still have overlooked some solution(s) other than the one(s) you found. Naturally, the check works fine here.

Example 2. We determine all solutions to the system of equations:

$$\left.\begin{array}{r} -y - 3z = -7 \\ 2x - y + 2z = 4 \\ -4x + 3y - 2z = -1 \end{array}\right\}. \tag{4.3}$$

We want a nonzero x-term (if possible) in the first equation, so we interchange the first two equations, to get:

$$\left.\begin{aligned} 2x - y + 2z &= 4 \\ - y - 3z &= -7 \\ -4x + 3y - z &= -1 \end{aligned}\right\}.$$

We now use the first equation to eliminate the x-term from the other equations. To do this we add twice the first equation to the third equation; we leave the second equation alone, since its x-term is already zero. We now have:

$$\left.\begin{aligned} 2x - y + 2z &= 4 \\ - y - 3z &= -7 \\ y + 3z &= 7 \end{aligned}\right\}.$$

We now use the second equation to eliminate y from the third equation. We do this by adding the second equation to the third equation, which gives:

$$\left.\begin{aligned} 2x - y + 2z &= 4 \\ - y - 3z &= -7 \\ 0 &= 0 \quad (!) \end{aligned}\right\}. \tag{4.4}$$

This system of equations is in echelon form, but has the rather interesting equation $0 = 0$. This prompts the following definition.

Definition. An equation $ax + by + cz = d$ is called *degenerate* if $a = b = c = 0$ (NB: we do allow $d \neq 0$ as well as $d = 0$). Otherwise it is *non-degenerate*.

There are two types of degenerate equations.

1. The degenerate equation $0 = 0$ (in 3 variables x, y, z) has as solutions $x = p$, $y = q$, $z = r$, for *all* real numbers p, q, r.

2. The degenerate equation $0 = d$, with $d \neq 0$, has *no* solutions. Note that the $=$ sign is being used in two different senses in the previous sentence: the first use relates two sides of an equation, and the second use is as equality. This may be confusing, but I am afraid you are going to have to get used to it.

Since the equation $0 = 0$ yields no restrictions whatsoever, we may discard it from the system of equations (4.4) to obtain:

$$\left.\begin{aligned} 2x - y + 2z &= 4 \\ - y - 3z &= -7 \end{aligned}\right\}. \tag{4.5}$$

This system of equations is in echelon form, and has no degenerate equations, and we solve this system of equations using the process of back substitution. The variable z can be any real number t, since z is not a leading variable in any of the equations of (4.5), where we define the term *leading variable* in the next chapter. Then the second equation gives $-y-3t=-7$, and so $y=7-3t$. Then the first equation gives $2x-(7-3t)+2t=4$, and so $2x=11-5t$, and hence $x=\frac{11}{2}-\frac{5}{2}t$.

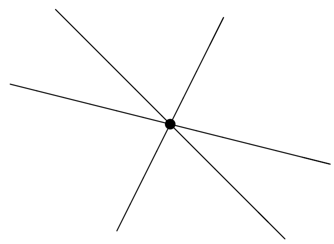
We have that $x=\frac{11}{2}-\frac{5}{2}t$, $y=7-3t$, $z=t$ is a solution for all real numbers t (this is infinitely[1] many solutions). Therefore, the intersection of the three planes defined by (4.3) is

$$\{\,(\tfrac{11}{2}-\tfrac{5}{2}t, 7-3t, t) : t \in \mathbb{R}\,\}.$$

This intersection is a line, with parametric equations:

$$\left.\begin{aligned} x &= \tfrac{11}{2}-\tfrac{5}{2}\lambda \\ y &= \ \ 7-3\lambda \\ z &= \qquad\ \lambda \end{aligned}\right\}.$$

When we take a cross-section through the configuration of planes defined by the original equations, we get a diagram like the following, where all the planes are perpendicular to the page.



The sceptic will wonder whether we have lost any information during the working of this example. We shall discover that the method we use preserves all the information contained in the original equations. Nevertheless, it is still prudent to check, for all real numbers t, that $(x,y,z)=(\frac{11}{2}-\frac{5}{2}t, 7-3t, t)$ is a solution to all of the original equations (4.3).

Example 2′. The equations here have the same left-hand sides as those in Example 2. However, their right-hand sides are different (I may have made a different alteration in lectures).

$$\left.\begin{aligned} - \ \ y - 3z &= -5 \\ 2x - \ \ y + 2z &= \ \ \ 4 \\ -4x + 3y - \ \ z &= -1 \end{aligned}\right\}. \tag{4.6}$$

The steps one must perform to bring the equations into echelon form are the same as in Example 2. In all cases, the left-hand sides should all be the same. However, the

[1] In 2010, the BBC broadcast a Horizon programme about Infinity, including contributions from Professor P. J. Cameron of our department. One thing we learned in the programme is that there are different sizes of infinite set. The sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ all have the same size, denoted $\aleph_0$ (the countably infinite cardinality), while $\mathbb{R}$ has a strictly bigger size, denoted $2^{\aleph_0}$. The symbol $\aleph$ is the first letter of the Hebrew alphabet, is called ’aleph, and traditionally stands for a glottal stop.

right-hand sides will differ. Echelonisation proceeds as follows. First swap the first and second equations:

$$\left.\begin{aligned} 2x - y + 2z &= 4 \\ - y - 3z &= -5 \\ -4x + 3y - z &= -1 \end{aligned}\right\}.$$

Add twice the first equation to the third:

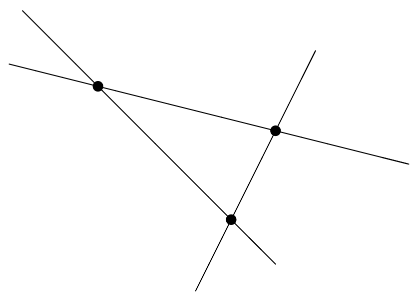
$$\left.\begin{aligned} 2x - y + 2z &= 4 \\ - y - 3z &= -5 \\ y + 3z &= 7 \end{aligned}\right\}.$$

Add the second equation to the third:

$$\left.\begin{aligned} 2x - y + 2z &= 4 \\ - y - 3z &= -5 \\ 0 &= 2 \end{aligned}\right\}.$$

But $0 = 2$ is a degenerate equation with *no* solutions, so the whole system of equations has no solutions, and thus the original system of equations has no solutions. There is no need to engage in back substitution for this example.

When we take a cross-section through the configuration of planes defined by the original equations, we get a diagram like the following, where all the planes are perpendicular to the page.



Checking all the solutions we obtain is vacuous in this example. A better bet is to follow through the echelonisation to try to figure out how to obtain the equation $0 = 2$ from the original equations (4.6). In this case, we find that we obtain $0 = 2$ by adding the first and third equations of (4.6) to twice the second equation of (4.6).

Example 3. We determine the intersection of the two planes defined by:

$$\left.\begin{aligned} x + 2y - z &= 2 \\ -2x - 4y + 2z &= 1 \end{aligned}\right\}. \tag{4.7}$$

Add twice the first equation to the second to get:

$$\left.\begin{aligned} x + 2y - z &= 2 \\ 0 &= 5 \end{aligned}\right\}.$$

But $0 = 5$ is a degenerate equation with *no* solutions, so the whole system of equations has no solutions, and thus the original system of equations has no solutions. So the intersection of the planes is $\varnothing = \{\}$, the empty set, which is the *only* set that has *no* elements. In this case, the original two planes were parallel, but not equal.

Example 4. [Not done in lectures.] If we have a system of $k = 0$ equations in unknowns x, y, z, then the solutions to this system of equations is $x = r$, $y = s$, $z = t$, where r, s, t can be any real numbers. The solution set is thus:

$$\{\,(r, s, t) : r, s, t \in \mathbb{R}\,\} = \mathbb{R}^3,$$

which corresponds to my earlier assertion that the intersection of 0 planes in $\mathbb{R}^3$ is the whole of $\mathbb{R}^3$.

4.3 Notes

We have been solving systems of linear equations by employing two basic types of operations on these equations to bring them into an easy-to-solve form called *echelon form*. These operations are:

(A) adding a multiple of one equation to another;

(I) interchanging two equations; and

(M) [not used by us] multiplying an equation by a *nonzero* number.

These are called *elementary operations* on the system of linear equations, and the corresponding operations on matrices (we define matrices later) are called *elementary row operations*.

These elementary operations (including (M)) are all invertible, and as a consequence *never* change the set of solutions of a system of linear equations (see Coursework 4). It is for this reason that we kept writing down equations we had seen previously, and not just the new equations we had found. The whole system of linear equations is important, and if we did not keep track of the whole system this then we might lose some information on the way and inadvertently deduce more solutions to our equations than the original equations had.

One should be careful how one annotates row operations. Please bear in mind that we are operating on systems of equations, which should thus be linked by a brace (you will lose marks for forgetting this). Writing $R_1 + 2R_2$ does not tell me the row operation you have performed. Does this mean add 2 copies of Row 2 to Row 1 (an operation you would never use)? In that case, you could write $R_1 \mapsto R_1 + 2R_2$ or $R_1' = R_1 + 2R_2$. Or does $R_1 + 2R_2$ replace Row 2? (This is not one of our basic operations, but I have still seen it in work I had to mark.) During Gaußian elimination, failure to indicate the row operations used, or indicating them ambiguously, is also liable to lose marks.