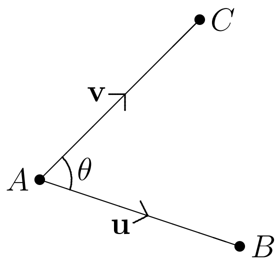
Chapter 3

The Scalar Product

The scalar product is a way of multiplying two vectors to produce a scalar (real number). Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors represented by $\overrightarrow{AB}$ and $\overrightarrow{AC}$.



We define the angle between $\mathbf{u}$ and $\mathbf{v}$ to be the angle θ (in radians) between $\overrightarrow{AB}$ and $\overrightarrow{AC}$, with $0 \leqslant \theta \leqslant \pi$. A handy chart for converting between degrees and radians is given below.

radians	0	$\frac{\pi}{180}$	$\frac{\pi}{12}$	$\frac{\pi}{10}$	$\frac{\pi}{6}$	$\frac{\pi}{5}$	$\frac{\pi}{4}$	1	$\frac{\pi}{3}$	$\frac{2\pi}{5}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π	2π
degrees	0	1	15	18	30	36	45	$\frac{180}{\pi} \approx 57.3$	60	72	90	120	135	180	360

Definition 3.1. The *scalar product* (or *dot product*) of $\mathbf{u}$ and $\mathbf{v}$ is denoted $\mathbf{u}\cdot\mathbf{v}$, and is defined to be 0 if either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$. If both $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$, we define $\mathbf{u}\cdot\mathbf{v}$ by

$$\mathbf{u}\cdot\mathbf{v} := |\mathbf{u}||\mathbf{v}| \cos\theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. (Note that I have had to specify what θ is in the definition itself; you <u>**must**</u> do the same.) We say that $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* if $\mathbf{u}\cdot\mathbf{v} = 0$.

Note that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ or the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\frac{\pi}{2}$. (This includes the case $\mathbf{u} = \mathbf{v} = \mathbf{0}$.)

Note. Despite the notation concealing this fact somewhat, the scalar product is a ***function***. Its codomain (and range) is $\mathbb{R}$, and its domain is the set of ordered pairs of (free) vectors. As usual, we must make sure that the function is defined (in a unique manner) for ***all*** elements of the domain, and this includes those pairs having the zero vector in one or both positions.

3.1 The scalar product using coördinates

Theorem 3.2. Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$. Then $\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$.

Proof. If $\mathbf{u} = \mathbf{0}$ (in which case $u_1 = u_2 = u_3 = 0$) or $\mathbf{v} = \mathbf{0}$ (in which case $v_1 = v_2 = v_3 = 0$) we have $\mathbf{u}\cdot\mathbf{v} = 0 = u_1v_1 + u_2v_2 + u_3v_3$, as required.

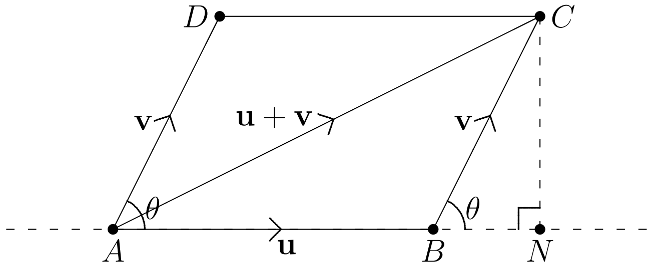
Now suppose that $\mathbf{u}, \mathbf{v} \neq 0$, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. We calculate $|\mathbf{u}+\mathbf{v}|^2$ in two different ways. Firstly we use coördinates.

$$\begin{aligned}|\mathbf{u}+\mathbf{v}|^2 = \left|\begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{pmatrix}\right|^2 &= (u_1+v_1)^2 + (u_2+v_2)^2 + (u_3+v_3)^2 \\ &= u_1^2 + 2u_1v_1 + v_1^2 + u_2^2 + 2u_2v_2 + v_2^2 + u_3^2 + 2u_3v_3 + v_3^2 \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2(u_1v_1 + u_2v_2 + u_3v_3),\end{aligned}$$

that is:

$$|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2(u_1v_1 + u_2v_2 + u_3v_3). \tag{3.1}$$

Our second way to do this is geometrical. Pick a point A, and consider the parallelogram $ABCD$, where $\overrightarrow{AB}$ represents $\mathbf{u}$ and $\overrightarrow{AD}$ represents $\mathbf{v}$. Thus $\overrightarrow{BC}$ represents $\mathbf{v}$, and so $\overrightarrow{AC}$ represents $\mathbf{u}+\mathbf{v}$ by the Triangle Rule. Drop a perpendicular from C to the line through A and B, meeting the said line at N, and let M be an arbitrary point on the line through A and B strictly to 'right' of B (i.e. when traversing the line through A and B in a certain direction we encounter the points in the order A, B, M). Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$ (i.e. θ is the size of angle BAD). A result from Euclidean geometry states that the angle MBC also has size θ. The following diagram has all this information.



(Note that this diagram is drawn with $0 < \theta < \frac{\pi}{2}$. If $\theta = \frac{\pi}{2}$ then $N = B$, and if $\theta > \frac{\pi}{2}$ then N lies to the 'left' of B, probably between A and B, but possibly even to the 'left' of A.) We have $|\overrightarrow{AN}| = |(|\mathbf{u}| + |\mathbf{v}|\cos\theta)|$, even when $\theta \geqslant \frac{\pi}{2}$, and even when N is to the 'left' of A. We also have that $|\overrightarrow{CN}| = |\mathbf{v}||\sin\theta|$. Applying Pythagoras (which is fine here even when $\theta \geqslant \frac{\pi}{2}$), and using the fact that $|a|^2 = a^2$ whenever $a \in \mathbb{R}$, we obtain:

$$\begin{aligned}|\mathbf{u}+\mathbf{v}|^2 = |\overrightarrow{AC}|^2 = |\overrightarrow{AN}|^2 + |\overrightarrow{CN}|^2 &= (|\mathbf{u}| + |\mathbf{v}|\cos\theta)^2 + (|\mathbf{v}|\sin\theta)^2 \\ &= |\mathbf{u}|^2 + 2|\mathbf{u}||\mathbf{v}|\cos\theta + |\mathbf{v}|^2(\cos\theta)^2 + |\mathbf{v}|^2(\sin\theta)^2 \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2(\cos^2\theta + \sin^2\theta) + 2\mathbf{u}\cdot\mathbf{v}.\end{aligned}$$

Here $\cos^2\theta$ means $(\cos\theta)^2$ and $\sin^2\theta$ means $(\sin\theta)^2$. Using the standard identity that $\cos^2\theta + \sin^2\theta = 1$ for all θ, we obtain:

$$|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2\mathbf{u}\cdot\mathbf{v}. \tag{3.2}$$

Comparing Equations 3.1 and 3.2 gives us the result. $\square$

Note that if $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ then $\mathbf{u}\cdot\mathbf{u} = u_1^2 + u_2^2 + u_3^2 = |\mathbf{u}|^2$ (even when $\mathbf{u} = \mathbf{0}$).

Example. We determine $\cos\theta$, where θ is the angle between $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$. We have $|\mathbf{u}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$ and $|\mathbf{v}| = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$, along with:

$$\mathbf{u}\cdot\mathbf{v} = 2\times 1 + (-1)\times 2 + 1\times(-3) = 2 - 2 - 3 = -3.$$

The formula $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$ gives $-3 = \sqrt{6}\sqrt{14}\cos\theta = 2\sqrt{21}\cos\theta$, the last step being since $\sqrt{6}\sqrt{14} = \sqrt{2}\sqrt{3}\sqrt{2}\sqrt{7} = 2\sqrt{21}$. Thus we get:

$$\cos\theta = \frac{-3}{2\sqrt{21}} = -\frac{1}{2}\sqrt{\frac{3}{7}}.$$

(The last equality was obtained by cancelling a factor of $\sqrt{3}$ from the numerator and denominator. There is no need to do this if it does not make the fraction 'neater', and here I do not think it does.)

Note. The following is an example of totally unacceptable working when calculating a dot product.

$$\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1\times 3 \\ (-1)\times(-2) \\ (-2)\times 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} = 3 + 2 + (-2) = 3.$$

This is because the first and third so-called equalities are nothing of the sort. The first is trying to equate a scalar (LHS) with a vector (RHS), while the third tries to equate a vector with a scalar. The above has ***TWO*** errors, and we shall simply mark such stuff as being wrong.

3.2 Properties of the scalar product

Let $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$ and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. From the definition $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$ of the scalar product we observe that:

- if $0 \leqslant \theta < \frac{\pi}{2}$ then $\mathbf{u}\cdot\mathbf{v} > 0$;
- if $\theta = \frac{\pi}{2}$ then $\mathbf{u}\cdot\mathbf{v} = 0$; and
- if $\frac{\pi}{2} < \theta \leqslant \pi$ then $\mathbf{u}\cdot\mathbf{v} < 0$.

Moreover,

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}.$$

Now let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be any vectors. Then:

1. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$;
2. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = (\mathbf{u}\cdot\mathbf{v}) + (\mathbf{u}\cdot\mathbf{w})$;
3. $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = (\mathbf{u}\cdot\mathbf{w}) + (\mathbf{v}\cdot\mathbf{w})$;
4. $\mathbf{u}\cdot(\alpha\mathbf{v}) = \alpha(\mathbf{u}\cdot\mathbf{v}) = (\alpha\mathbf{u})\cdot\mathbf{v}$ for all scalars α;
5. $\mathbf{u}\cdot(-\mathbf{v}) = (-\mathbf{u})\cdot\mathbf{v} = -(\mathbf{u}\cdot\mathbf{v})$; and
6. $(-\mathbf{u})\cdot(-\mathbf{v}) = \mathbf{u}\cdot\mathbf{v}$.

There is however no (non-vacuous) associative law for the dot product. This is because neither of the quantities $(\mathbf{u}\cdot\mathbf{v})\cdot\mathbf{w}$ and $\mathbf{u}\cdot(\mathbf{v}\cdot\mathbf{w})$ is defined. (In both cases, we are trying to form the dot product of a vector and a scalar in some order, and in neither order does such a product exist.)

Each of the above equalities can be proved by using Theorem 3.2, which expresses the dot product in terms of coördinates. To prove (1) we observe that:

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = v_1u_1 + v_2u_2 + v_3u_3 = \mathbf{v}\cdot\mathbf{u}.$$

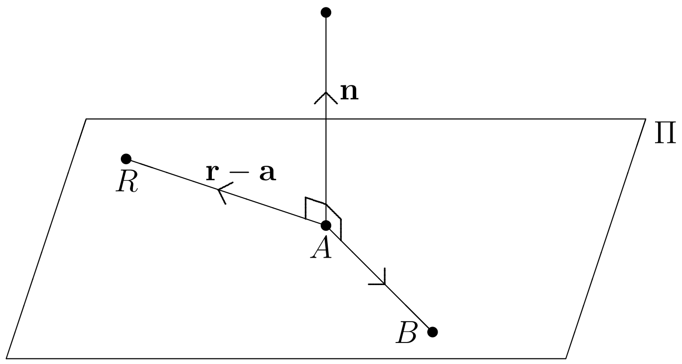
In order to prove the equality $\mathbf{u}\cdot(\alpha\mathbf{v}) = \alpha(\mathbf{u}\cdot\mathbf{v})$ of (4) we observe the following.

$$\begin{aligned}\mathbf{u}\cdot(\alpha\mathbf{v}) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}\cdot\begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ \alpha v_3 \end{pmatrix} &= u_1(\alpha v_1) + u_2(\alpha v_2) + u_3(\alpha v_3) \\ &= \alpha(u_1v_1) + \alpha(u_2v_2) + \alpha(u_3v_3) \\ &= \alpha(u_1v_1 + u_2v_2 + u_3v_3) = \alpha(\mathbf{u}\cdot\mathbf{v}).\end{aligned}$$

The proofs of the rest of these equalities are left as exercises.

3.3 Equation of a plane

Let $\mathbf{n}$ be a vector and Π be a plane. We say that $\mathbf{n}$ is *orthogonal* to Π (or Π is orthogonal to $\mathbf{n}$) if for all points A, B on Π, we have that $\mathbf{n}$ is orthogonal to the vector represented by $\overrightarrow{AB}$. We also say that $\mathbf{n}$ is a *normal* (or normal vector) to Π, hence the notation $\mathbf{n}$.



Suppose that $\mathbf{n} \neq \mathbf{0}$, A is a point, and we wish to determine an equation of the (unique) plane Π that is orthogonal to $\mathbf{n}$ and contains A. Now a point R, with position vector $\mathbf{r}$, is on Π exactly when $\overrightarrow{AR}$ represents a vector orthogonal to $\mathbf{n}$, that is when $(\mathbf{r}-\mathbf{a})\cdot\mathbf{n} = 0$, where $\mathbf{a}$ is the position vector of $\mathbf{a}$. Equivalently, we have $\mathbf{r}\cdot\mathbf{n} - \mathbf{a}\cdot\mathbf{n} = 0$, which gives:

$$\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n},$$

a *vector equation* of the plane Π, where $\mathbf{r}$ is the position vector of an arbitary point on Π, $\mathbf{a}$ is the position vector of a fixed point on Π, and $\mathbf{n}$ is a nonzero vector orthogonal to Π. In coördinates, we let

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \quad \text{and} \quad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Then the point (x, y, z) is on Π exactly when:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

that is, when

$$n_1x + n_2y + n_3z = d,$$

where $d = a_1n_1 + a_2n_2 + a_3n_3$. This is a *Cartesian equation* of the plane Π.

Example. We find a Cartesian equation for the plane through $A = (2, -1, 3)$ and orthogonal to $\mathbf{n} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}$. A vector equation is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}$, which gives rise to the Cartesian equation $-2x + 3y + 5z = 8$.

Example. The equation $2x-y+3z=6$ specifies the plane Π orthogonal to $\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and containing the point $(1,-1,1)$. This is because we can write the equation as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix},$$

which has the form $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$ for suitable vectors $\mathbf{r}$ and $\mathbf{a}$. The point $(1,2,3)$ is not on Π since $2\times 1 + (-1)\times 2 + 3\times 3 = 2-2+9 = 9 \neq 6$. The point $(1,2,2)$ is on Π since $2\times 1 + (-1)\times 2 + 3\times 2 = 2-2+6 = 6$.

Note that the coördinates of $\mathbf{n}$ can always be taken to be the coefficients of x, y, z in the Cartesian equation. (It is valid to multiply such an $\mathbf{n}$ by any nonzero scalar, but must ensure we do the corresponding operations to the right-hand sides of any equations we use. Thus both $2x-y+3x=6$ and $-4x+2y-6x=-12$ are Cartesian equations of the plane Π in the second example above.) Finding a point on Π is harder. A sensible strategy is to set two of x, y, z to be zero (where the coefficient of the third is nonzero). Here setting $x=y=0$ gives $3z=6$, whence $z=2$, so that $(0,0,2)$ is on Π. Setting $x=z=0$ gives $y=-6$, so that $(0,-6,0)$ is on Π, and setting $y=z=0$ gives $x=3$, so that $(3,0,0)$ is on Π. (This sensible strategy does not find the point $(1,-1,1)$ that is on Π.)

In the case of the plane Π' with equation $x+y=1$, setting $x=z=0$ gives the point $(0,1,0)$ on Π', while setting $y=z=0$ gives the point $(1,0,0)$ on Π'. But if we set $x=y=0$, we end up with the equation $0=1$, which has no solutions for z, so we do not find a point here.

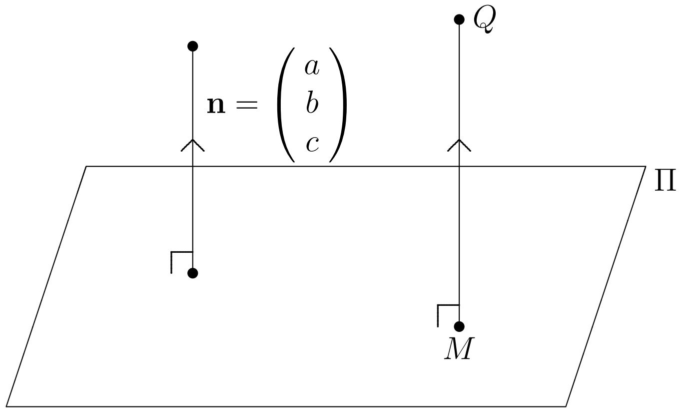
Note. [Not lectured.] Another form of a vector equation for a plane, corresponding to the vector equation for a line is as follows. Take any 3 points A, B, C on Π such that A, B, C are not on the same line. Let A, B, C have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively. Then a vector equation for Π is:

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b}-\mathbf{a}) + \mu(\mathbf{c}-\mathbf{a}),$$

where λ and μ range (independently) over the whole of $\mathbb{R}$.

3.4 The distance from a point to a plane

Let Π be the plane having equation $ax+by+cz=d$, so that Π is orthogonal to $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \neq \mathbf{0}$. Let Q be a point, and let M be the point on Π closest to Q.



Let $\mathbf{m}$, $\mathbf{q}$ be the position vectors of M, Q. Then the vector $\mathbf{q}-\mathbf{m}$ represented by $\overrightarrow{MQ}$ is orthogonal to Π, and so (since we are in just 3 dimensions) $\mathbf{q}-\mathbf{m}$ is a scalar multiple of $\mathbf{n}$, that is $\mathbf{q}-\mathbf{m} = \lambda\mathbf{n}$ for some scalar $\lambda \in \mathbb{R}$. Therefore:

$$(\mathbf{q}-\mathbf{m})\cdot\mathbf{n} = (\lambda\mathbf{n})\cdot\mathbf{n} = \lambda(\mathbf{n}\cdot\mathbf{n}) = \lambda|\mathbf{n}|^2,$$

and so $\mathbf{q}\cdot\mathbf{n}-\mathbf{m}\cdot\mathbf{n} = \lambda|\mathbf{n}|^2$. But $\mathbf{m}$ is on Π, which has equation $\mathbf{r}\cdot\mathbf{n} = d$, and so $\mathbf{m}\cdot\mathbf{n} = d$. Therefore $\mathbf{q}\cdot\mathbf{n} - d = \lambda|\mathbf{n}|^2$, and thus:

$$\lambda = \frac{\mathbf{q}\cdot\mathbf{n} - d}{|\mathbf{n}|^2}$$

But the distance from Q to Π (which is the distance from Q to M, where M is the closest point on Π to Q) is in fact $|\overrightarrow{MQ}| = |\lambda\mathbf{n}| = |\lambda||\mathbf{n}|$, that is:

$$\left|\frac{\mathbf{q}\cdot\mathbf{n} - d}{|\mathbf{n}|^2}\right||\mathbf{n}| = \frac{|\mathbf{q}\cdot\mathbf{n} - d|}{|\mathbf{n}|}.$$

If one looks at other sources one may see a superficially dissimilar formula for this distance. To obtain this, we let P be *any* point on Π, and let $\mathbf{p}$ be the position vector of $\mathbf{p}$, so that $\mathbf{p}\cdot\mathbf{n} = d$. Thus $\mathbf{q}\cdot\mathbf{n} - d = \mathbf{q}\cdot\mathbf{n} - \mathbf{p}\cdot\mathbf{n} = (\mathbf{q}-\mathbf{p})\cdot\mathbf{n}$. Therefore the distance can also be expressed as:

$$\frac{|(\mathbf{q}-\mathbf{p})\cdot\mathbf{n}|}{|\mathbf{n}|} = \left|(\mathbf{q}-\mathbf{p})\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\mathrm{free}(\overrightarrow{PQ})\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right| = |(\mathbf{q}-\mathbf{p})\cdot\hat{\mathbf{n}}|,$$

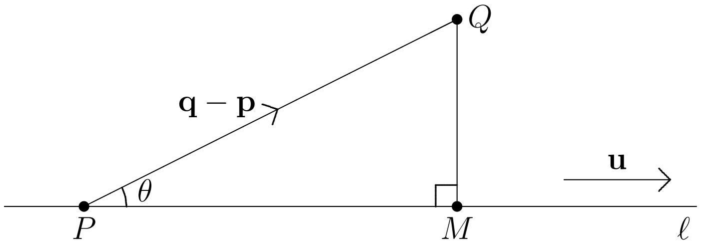
where $\mathrm{free}(\overrightarrow{PQ}) = \mathbf{q}-\mathbf{p}$ is the (free) vector represented by $\overrightarrow{PQ}$, and $\hat{\mathbf{n}}$ is the unit vector in the direction of $\mathbf{n}$.

Example. We find the distance of $(3,-2,4)$ from the plane defined by $2x+3y-5z = 7$. With our notation we have $\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$, $d = 7$, $\mathbf{q} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$, and so the distance is

$$\frac{|\mathbf{q}\cdot\mathbf{n} - d|}{|\mathbf{n}|} = \frac{|(6-6-20)-7|}{\sqrt{2^2+3^2+(-5)^2}} = \frac{27}{\sqrt{38}}.$$

3.5 The distance from a point to a line

Let ℓ be the line with (vector) equation $\mathbf{r} = \mathbf{p} + \lambda\mathbf{u}$, where $\mathbf{u} \neq \mathbf{0}$, and let Q be a point with position vector $\mathbf{q}$. If $Q = P$ (where P has position vector P), then Q is on ℓ, and the distance between Q and ℓ is 0. Else we drop a normal from Q to ℓ meeting ℓ at the point M.



The distance from Q to ℓ is $|\overrightarrow{MQ}|$, that is $|\mathbf{q} - \mathbf{m}|$, is easily seen from the diagram to be $|\overrightarrow{PQ}|\sin\theta$, where θ is the angle between $\mathbf{q} - \mathbf{p}$ (the vector that $\overrightarrow{PQ}$ represents) and the vector $\mathbf{u}$ (which is the direction [up to opposite] of the line ℓ). [Pedants should recall that $\mathbf{u} \neq \mathbf{0}$, and also note that $\sin\theta \geqslant 0$, since $0 \leqslant \theta \leqslant \pi$.] For now we content ourselves with noting that this distance is $|\mathbf{q} - \mathbf{p}|\sin\theta$ (which 'morally' applies even when $\mathbf{q} = \mathbf{p}$). When we encounter the cross product, we shall be able to express this distance as $|(\mathbf{q} - \mathbf{p}) \times \mathbf{u}|/|\mathbf{u}|$. Note that $\sin\theta$ can be calculated using dot products, since $\cos\theta$ can be so calculated, and we have $\sin\theta = \sqrt{1 - \cos^2\theta}$. On calculating $|\mathbf{q} - \mathbf{p}|\sin\theta$, we find that the distance from Q to ℓ is

$$\frac{\sqrt{|\mathbf{q} - \mathbf{p}|^2|\mathbf{u}|^2 - ((\mathbf{q} - \mathbf{p})\cdot\mathbf{u})^2}}{|\mathbf{u}|},$$

a formula which applies even when $\mathbf{q} = \mathbf{p}$. In the case when $|\mathbf{u}| = 1$ the above formula simplifies to $\sqrt{|\mathbf{q} - \mathbf{p}|^2 - ((\mathbf{q} - \mathbf{p})\cdot\mathbf{u})^2}$.

Exercise. Use methods from Calculus I to minimise the distance from R to Q, where R, the typical point on ℓ, has position vector $\mathbf{r}$ with $\mathbf{r} = \mathbf{p} + \lambda\mathbf{u}$. Show that this minimum agrees with the distance from Q to ℓ given above. Hint: The quantity $|\mathbf{r} - \mathbf{q}|$ is always at least 0. So $|\mathbf{r} - \mathbf{q}|$ is minimal precisely when $|\mathbf{r} - \mathbf{q}|^2$ is minimal. But $|\mathbf{r} - \mathbf{q}|^2 = (\mathbf{r} - \mathbf{q})\cdot(\mathbf{r} - \mathbf{q})$, and this is easier to deal with.