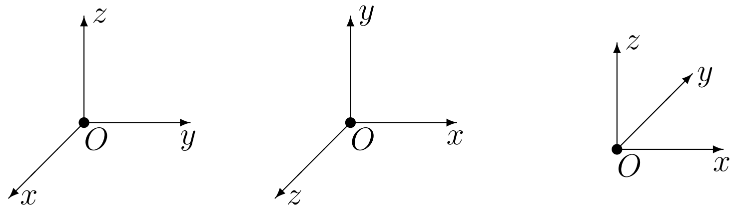
Chapter 2

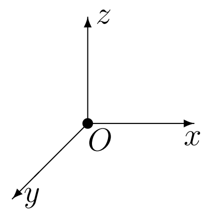
Cartesian Coördinates

The adjective *Cartesian* above refers to René Descartes (1596–1650), who was the first to coördinatise the plane as ordered pairs of real numbers, which provided the first systematic link between Euclidean geometry and algebra.

Choose an origin O in 3-space, and choose (three) mutually perpendicular axes through O, which we shall label as the x-, y- and z-axes. The x-axis, y-axis and z-axis form a right-handed system if they can be rotated to look like one of the following (which can all to rotated to look like the others).



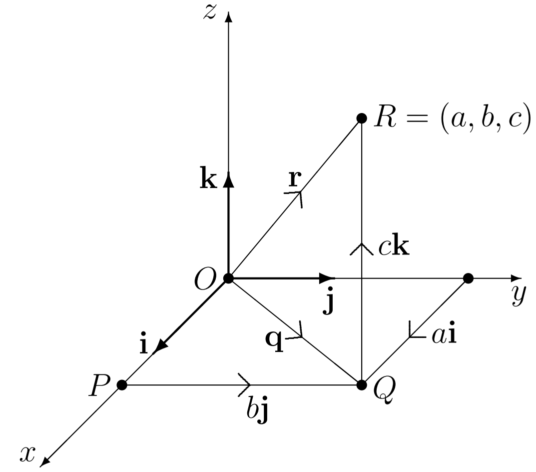
A left-handed system can be rotated to look like the following.



Swapping two axes or reversing the direction of one (or three) of the axes changes the handedness of the system. It is possible to make the shape of a right-handed system using our right-hand, with the thumb (pointing) along the x-axis, first [index] finger along the y-axis, and second [middle] finger along the z-axis. You should curl the other two fingers of your hand into you palm when you do this. (Unfortunately, it is possible, though much harder, to make the shape of a left-handed system using your right hand, but if you can make such a configuration it should be much more uncomfortable!) If you use your left hand, you should end up with a left-handed system.

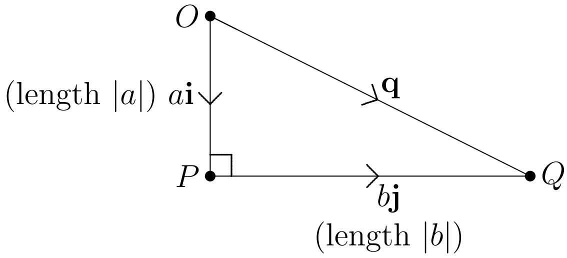
We let **i**, **j** and **k** denote vectors of length 1 in the directions of the x-, y- and z-axes respectively.

Let R be the point whose coördinates are (a, b, c), and let $\mathbf{r}$ be the position vector of R. Then $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. See the diagram below.

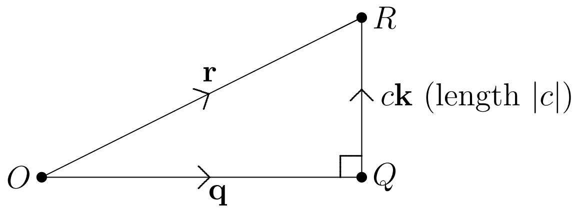


Let $\mathbf{q} = a\mathbf{i} + b\mathbf{j}$ be the position vector of the point Q. Then applying Pythagoras's Theorem to the right-angled triangle OPQ we get:

$$|\mathbf{q}| = |\overrightarrow{OQ}| = \sqrt{|\overrightarrow{OP}|^2 + |\overrightarrow{PQ}|^2} = \sqrt{|a|^2 + |b|^2} = \sqrt{a^2 + b^2}.$$



We also have the right-angled triangle OQR.



Applying Pythagoras's Theorem to triangle OQR gives:

$$|\mathbf{r}| = |\overrightarrow{OR}| = \sqrt{|\overrightarrow{OQ}|^2 + |\overrightarrow{QR}|^2} = \sqrt{|\mathbf{q}|^2 + |c|^2} = \sqrt{a^2 + b^2 + c^2}.$$

To summarise: If R is a point having coördinates (a, b, c), then the position vector of R is $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, which has length $\sqrt{a^2 + b^2 + c^2}$.

Notation. We write $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ for the vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

2.1 Sums and scalar multiples using coördinates

Now let $\mathbf{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$ and let α be a scalar. Then, using the rules for vector addition and scalar multiplication (sometimes multiple times per line) we get:

$$\begin{aligned}
\mathbf{u}+\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} &= (a\mathbf{i}+b\mathbf{j}+c\mathbf{k}) + (d\mathbf{i}+e\mathbf{j}+f\mathbf{k}) \\
&= (a\mathbf{i}+d\mathbf{i}) + (b\mathbf{j}+e\mathbf{j}) + (c\mathbf{k}+f\mathbf{k}) \\
&= \quad (a+d)\mathbf{i} + (b+e)\mathbf{j} + (c+f)\mathbf{k} \quad = \begin{pmatrix} a+d \\ b+e \\ c+f \end{pmatrix},
\end{aligned}$$

along with

$$\begin{aligned}
\alpha\mathbf{u} = \alpha \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= \alpha(a\mathbf{i}+b\mathbf{j}+c\mathbf{k}) \\
&= \alpha(a\mathbf{i}) + \alpha(b\mathbf{j}) + \alpha(c\mathbf{k}) \\
&= (\alpha a)\mathbf{i} + (\alpha b)\mathbf{j} + (\alpha c)\mathbf{k} = \begin{pmatrix} \alpha a \\ \alpha b \\ \alpha c \end{pmatrix},
\end{aligned}$$

and

$$-\mathbf{u} = (-1)\mathbf{u} = (-1)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -a \\ -b \\ -c \end{pmatrix}.$$

Example. Let $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$. Then

$$3\mathbf{u} - 4\mathbf{v} = 3\mathbf{u} + (-4)\mathbf{v} = 3\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + (-4)\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 0 \end{pmatrix} + \begin{pmatrix} -12 \\ -20 \\ 4 \end{pmatrix} = \begin{pmatrix} -6 \\ -23 \\ 4 \end{pmatrix}.$$

2.2 Unit vectors

Definition 2.1. A *unit vector* is a vector of length 1.

For example, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors. Let $\mathbf{r}$ be any nonzero vector, and define:

$$\hat{\mathbf{r}} := \left(\frac{1}{|\mathbf{r}|}\right)\mathbf{r}.$$

(Note that $|\mathbf{r}| > 0$, so that $\frac{1}{|\mathbf{r}|}$ and hence $\hat{\mathbf{r}}$ exist.) But we also have $\frac{1}{|\mathbf{r}|} > 0$, and thus $|\hat{\mathbf{r}}| = |\frac{1}{|\mathbf{r}|}||\mathbf{r}| = \frac{1}{|\mathbf{r}|}|\mathbf{r}| = 1$, so that $\hat{\mathbf{r}}$ is the unit vector in the same direction as $\mathbf{r}$. (There is only one unit vector in the same direction as $\mathbf{r}$.)

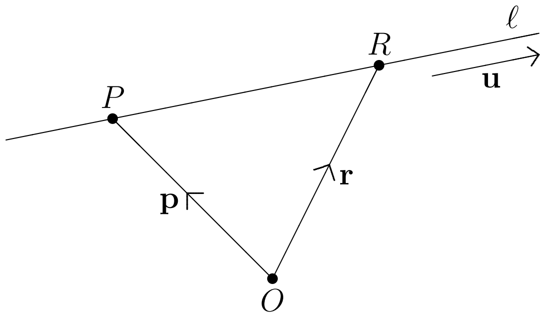
Example. Let $\mathbf{r} = \begin{pmatrix} -1 \\ 5 \\ -4 \end{pmatrix}$. Then $|\mathbf{r}| = \sqrt{(-1)^2 + 5^2 + (-4)^2} = \sqrt{42}$. Therefore

$$\hat{\mathbf{r}} = \left(\frac{1}{|\mathbf{r}|}\right)\mathbf{r} = \frac{1}{\sqrt{42}}\begin{pmatrix} -1 \\ 5 \\ -4 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{42} \\ 5/\sqrt{42} \\ -4/\sqrt{42} \end{pmatrix}.$$

If we want the unit vector in the *opposite* direction to $\mathbf{r}$, this is simply $-\hat{\mathbf{r}}$, and if we want the vector of length 7 in the opposite direction to $\mathbf{r}$, this is $-7\hat{\mathbf{r}}$.

2.3 Equations of lines

Let ℓ be the line through the point P in the direction of the *nonzero* vector $\mathbf{u}$.



Now a point R is on the line ℓ if and only if $\overrightarrow{PR}$ represents a scalar multiple of $\mathbf{u}$. Let $\mathbf{p}$ and $\mathbf{r}$ be the position vectors of P and R respectively. Then $\overrightarrow{PR}$ represents $\mathbf{r} - \mathbf{p}$, and so R is on ℓ if and only if $\mathbf{r} - \mathbf{p} = \lambda\mathbf{u}$ for some real number λ; equivalently $\mathbf{r} = \mathbf{p} + \lambda\mathbf{u}$ for some real number λ. We thus get the *vector equation* of the line ℓ:

$$\mathbf{r} = \mathbf{p} + \lambda\mathbf{u} \qquad (\lambda \in \mathbb{R}),$$

where $\mathbf{p}$ and $\mathbf{u}$ are constants and $\mathbf{r}$ is a variable (depending on λ) which denotes the position of a general point on ℓ.

Moving to coördinates, we let $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$. Then:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{r} = \mathbf{p} + \lambda\mathbf{u} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + \lambda\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} p_1 + \lambda u_1 \\ p_2 + \lambda u_2 \\ p_3 + \lambda u_3 \end{pmatrix},$$

from which we get the *parametric equations* of the line ℓ, namely:

$$\left.\begin{aligned} x &= p_1 + \lambda u_1 \\ y &= p_2 + \lambda u_2 \\ z &= p_3 + \lambda u_3 \end{aligned}\right\}.$$

Assuming that $u_1, u_2, u_3 \neq 0$, we may eliminate λ to get:

$$\frac{x - p_1}{u_1} = \frac{y - p_2}{u_2} = \frac{z - p_3}{u_3},$$

which are the *Cartesian equations* of the line ℓ. (Each of these fractions is equal to λ.) The following should tell you how to get the Cartesian equations of ℓ when one or two of the u_i are zero (they cannot all be zero). If $u_1 = 0$ and $u_2, u_3 \neq 0$ then the Cartesian equations are

$$x = p_1, \ \frac{y - p_2}{u_2} = \frac{z - p_3}{u_3},$$

and if $u_1 = u_2 = 0$, $u_3 \neq 0$, the Cartesian equations are $x = p_1$, $y = p_2$ (with no mention of z anywhere).

Example. As an example, we determine the vector, parametric and Cartesian equations of the line ℓ through the point $(3, -1, 2)$ in the direction of the vector $\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$.

The vector equation is:

$$\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}.$$

The parametric equations are:

$$\left.\begin{aligned} x &= 3 - 2\lambda \\ y &= -1 + \lambda \\ z &= 2 + 4\lambda \end{aligned}\right\}.$$

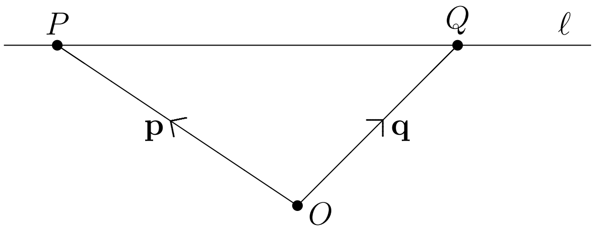
And the Cartesian equations are:

$$\frac{x - 3}{-2} = \frac{y + 1}{1} = \frac{z - 2}{4}.$$

Is the point $(7, -3, -6)$ on ℓ? Yes, since the vector equation is satisfied with $\lambda = -2$ (this value of λ can be determined from the parametric equations). What about the point $(1, 1, 1)$? No, because the Cartesian equations are not satisfied: $1 = \frac{1-3}{-2} \neq \frac{1+1}{1} = 2$. Alternatively, we can look at the parametric equations: the first would give $\lambda = 1$, and the second would give $\lambda = 2$, an inconsistency.

2.3.1 The line determined by two distinct points

Suppose we are given two points P and Q on a line ℓ, with $P \neq Q$, and we want to determine (say) a vector equation for ℓ. Suppose P has position vector $\mathbf{p}$ and Q has position vector $\mathbf{q}$.



Then ℓ is a line through P in the direction of $\overrightarrow{PQ}$, and thus in the direction of $\mathbf{q} - \mathbf{p}$ (the vector that $\overrightarrow{PQ}$ represents). Therefore, a vector equation for ℓ is:

$$\mathbf{r} = \mathbf{p} + \lambda(\mathbf{q} - \mathbf{p}).$$

Parametric and Cartesian equations for ℓ can be derived from this vector equation in the usual way.