Chapter 1

Vectors

1.1 Introduction

The word *geometry* derives from the Ancient Greek word γεωμετρία, with a rough meaning of *geometry, land-survey*, though I prefer *earth measurement*. There are two elements in this word: γῆ (or Γῆ), meaning *Earth* (among other related words), and either μετρέω, *to measure, to count*, or μέτρον, *a measure*. Given this etymology, one should have a fair idea of what geometry is about. (Consult the handouts on the web for a copy of the Greek alphabet; the letter names should give a [very] rough guide to the pronunciations of the letters themselves.)

In this module, we are interested in lines, planes, and other geometrical objects in 3-dimensional space (and maybe spaces of other dimensions).

We shall introduce standard notation for some number systems.

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, the *natural numbers*. I always include 0 as a natural number; some people do not.

- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the *integers*. The notation comes from the German *Zahlen* meaning *numbers*.

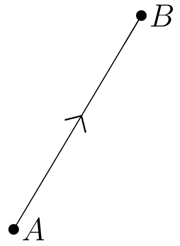
- $\mathbb{N}^+ = \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, the *positive integers*, and $\mathbb{N}_0 = \mathbb{Z}_{\geqslant 0} = \{0, 1, 2, 3, \ldots\}$, the *non-negative integers*. The word *positive* here means *strictly positive*, that is to say 0 is not considered to be a positive (or negative) number.

- $\mathbb{Q} = \{\, \frac{a}{b} : a, b \text{ are integers and } b \neq 0 \,\}$. The Q is the first letter of *quotient*.

- $\mathbb{R}$ denotes the set of 'real' numbers. Examples of real numbers are 2, $\frac{3}{5}$, $\sqrt{7}$ and π. Also, all decimal numbers, both terminating and not, are real numbers. In fact, each real number can be represented in decimal form, though this decimal is usually non-terminating and non-recurring. An actual definition of $\mathbb{R}$ is somewhat technical, and is deferred (for a long time). The set $\mathbb{R}$ is a very artificial construct and not very real at all.

1.2 Vectors

A *bound vector* is a bounded (and directed) line segment $\overrightarrow{AB}$ in 3-dimensional [real] space, which we shorten to 3-space and denote by $\mathbb{R}^3$, where A and B are points in this space. (There is no particular reason, except familiarity, to restrict ourselves to 3-space, and one often works in much higher dimensions, for example 196884-space.) We point out that the real world around us probably bears little resemblence to $\mathbb{R}^3$, despite the fact we are fondly imagining that it does.



Note that a bound vector $\overrightarrow{AB}$ is determined by (and determines) three things:

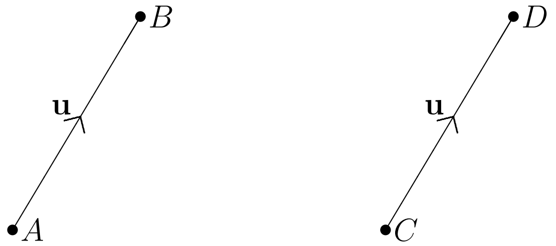
(i) its *length* or *magnitude*, denoted $|\overrightarrow{AB}|$,

(ii) its *direction* [provided $A \neq B$], and

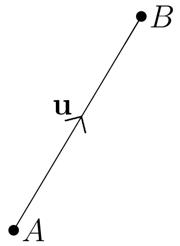
(iii) its *starting point*, which is A.

If $A = B$, then $\overrightarrow{AB} = \overrightarrow{AA}$ does not have a defined direction, and in this case $\overrightarrow{AB}$ is determined by its length, which is 0, and its starting point A.

If we ignore the starting point, and only care about the length and direction, we get the notion of a *free vector* (or simply *vector* in what follows). Thus $\overrightarrow{AB}$ and $\overrightarrow{CD}$ represent the same free vector if and only if they have the same length and direction.



We use $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, ... to denote free vectors (these would be underlined when hand-written: thus $\underline{u}, \underline{v}, \underline{w}, \ldots$), and draw



to mean that the free vector $\mathbf{u}$ is represented by $\overrightarrow{AB}$; that is, the length and direction of $\mathbf{u}$ are those of the bound vector $\overrightarrow{AB}$. We denote the length of $\mathbf{u}$ by $|\mathbf{u}|$.

Note. We should *never* write something like $\overrightarrow{AB} = \mathbf{u}$, tempting though it may be. The reason is that two objects on each side of the equal sign are different types of object (a bound vector versus a free vector), and it is always inappropriate to relate different types of object using the equality sign.

Note. The module text uses $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}, \dots$ for (free) vectors; this is perfectly standard in printed works. The previous lecturer used $\underline{u}, \underline{v}, \underline{w}, \dots$ (see the 2010 exam, for example), which is decidedly non-standard in print. The book also uses $\mathbf{AB}$ for the free vector represented by $\overrightarrow{AB}$, which we shall never use. Better notations for the free vector represented by $\overrightarrow{AB}$ are free($\overrightarrow{AB}$) or $[\overrightarrow{AB}]$, but we shall hardly ever use these either.[1]

Note. Each bound vector represents a unique free vector. Also, for each free vector $\mathbf{u}$ and for each point A there is a unique point B such that $\overrightarrow{AB}$ represents $\mathbf{u}$. This is a consequence of a bound vector being determined by its length, direction and starting point, and a free vector being determined by its length and direction only. Of course, a suitable (and annoying) modification must be made to the above when the zero vector (see below) is involved. We leave such a modification to the reader.

1.3 The zero vector

The *zero vector* is the (free) vector with zero length. Its direction is undefined. We denote the zero vector by $\mathbf{0}$ (or $\underline{0}$ in handwriting). It is represented by $\overrightarrow{AA}$, where A can be any point. (It is also represented by $\overrightarrow{DD}$, where D can be any point, and so on.)

1.4 Vector negation

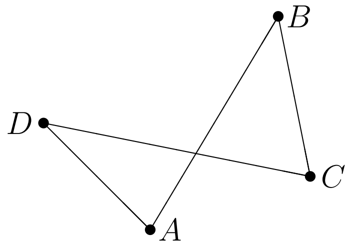
If $\mathbf{v}$ is a nonzero vector, then the negative of $\mathbf{v}$, denoted $-\mathbf{v}$, is the vector with the same length as $\mathbf{v}$ but opposite direction. We define $-\mathbf{0} := \mathbf{0}$. If $\overrightarrow{AB}$ represents $\mathbf{v}$ then $\overrightarrow{BA}$ represents $-\mathbf{v}$.

Note. Vector negation is a function from the set of free vectors to itself. It is therefore essential that it be defined for ***every*** element in the domain. That is, we must define the negative of every free vector. Note here the special treatment of the zero vector, which is not covered by the first sentence.

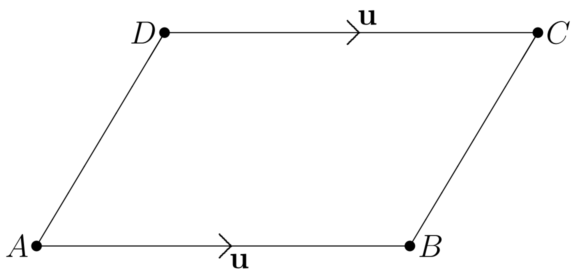
[1]Usually, $[\overrightarrow{AB}]$ would denote the equivalence class containing $\overrightarrow{AB}$. Here the relevant equivalence relation is that two bound vectors are equivalent if and only if they represent the same free vector. There is an obvious bijection between the set of these equivalence classes and the set of free vectors. See the module MTH4110: Mathematical Structures for definitions of equivalence relation and equivalence class.

1.5 Parallelograms

Suppose A, B, C, D are any points in 3-space. We obtain the figure $ABCD$ by joining A to B (by a [straight] line segment), B to C, C to D, and finally D to A. For example:



The figure $ABCD$ is called a *parallelogram* if $\overrightarrow{AB}$ and $\overrightarrow{DC}$ represent the same vector.



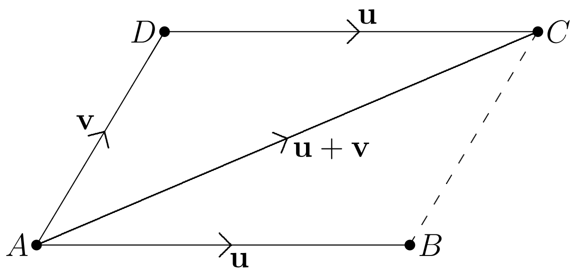
We note the following fact, which is really an axiom. We shall make some use of this later.

Fact (Parallelogram Axiom)**.** Note that $\overrightarrow{AB}$ and $\overrightarrow{DC}$ represent the same vector ($\mathbf{u}$ say) if and only if $\overrightarrow{AD}$ and $\overrightarrow{BC}$ represent the same vector ($\mathbf{v}$ say). We can have $\mathbf{u} = \mathbf{v}$, though more usually we will have $\mathbf{u} \neq \mathbf{v}$.

Note. We now see the folly of writing expressions like $\overrightarrow{AD} = \mathbf{v}$, where one side is a bound vector, and one side is a free vector. For example, in the above parallelogram, we would notice that $\overrightarrow{AB} = \mathbf{u} = \overrightarrow{DC}$ and deduce, using a well-known property of equality, that $\overrightarrow{AB} = \overrightarrow{DC}$. But this is nonsense in the general case (when $A \neq D$), since the vectors $\overrightarrow{AB}$ and $\overrightarrow{DC}$ have different starting points and are therefore *not* equal.

1.6 Vector addition

Now suppose that $\mathbf{u}$ and $\mathbf{v}$ are any vectors. Choose a point A. Further, assume that points B and D are chosen so that $\overrightarrow{AB}$ represents $\mathbf{u}$ and $\overrightarrow{AD}$ represents $\mathbf{v}$. (The points B and D are unique.) We extend the A, B, D-configuration to a parallelogram by choosing a point C (which is unique) such that $\overrightarrow{DC}$ represents $\mathbf{u}$, as in the diagram below. (Note that $\overrightarrow{BC}$ represents $\mathbf{v}$ by the Parallelogram Axiom.)



The *sum* of **u** and **v**, which we denote as $\mathbf{u}+\mathbf{v}$, is defined to be the vector represented by $\overrightarrow{AC}$.

Note that I have defined vector addition only for *free* vectors, *not* for bound vectors, so I do not wish to see you write things like $\overrightarrow{AB}+\overrightarrow{CD}=\cdots$.

1.7 Some notation

In lectures, you will often see abbreviations for various mathematical concepts. Some of these appear less often in printed texts. At least one of the symbols ($\forall$) was introduced around here. Most of these symbols can be negated.

- s.t. means 'such that'.
- $\forall$ means 'for all'.
- $\exists$ means 'there exists', while $\exists!$ means 'there exists unique'.
- $\nexists$ means 'there does not exist'.
- $a \in B$ means that the element a is a member of the set B.
- $a \notin B$ means that the element a is not a member of the set B.
- $A \subseteq B$ means that the set A is a subset of the set B (allows the case $A = B$).

1.8 Rules for vector addition

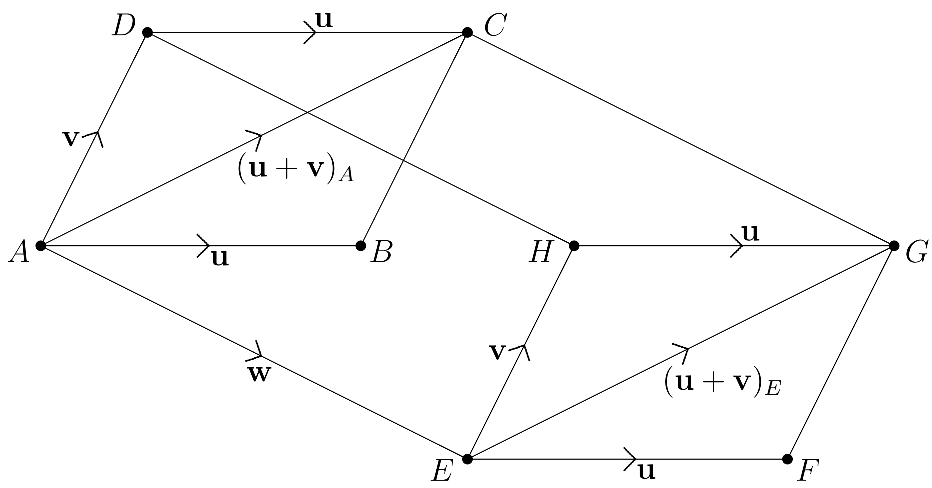
In the definition of $\mathbf{u}+\mathbf{v}$ you will notice the use of an arbitrary point A. When one encounters something like this, one is entitled to ask whether the definition depends on the point A or not. Mathematicians, being pedants, very often will ask such seemingly obvious questions. Temporarily, we shall use $(\mathbf{u}+\mathbf{v})_A$ to denote the value of $\mathbf{u}+\mathbf{v}$ obtained if the arbitrary point A was used in its definition. (There is no need to worry about the points B, C and D, since these are uniquely determined given **u**, **v** and A.)

We also take the opportunity in the following couple of pages to introduce terms such as *commutative*, *associative*, *identity*, *inverse* and *distributive*. You should meet these terms many times in your mathematical career. In the lectures we proved Theorems 1.2

and 1.3 before Theorem 1.1. The box at the end of the proofs is the end-of-proof symbol. One can write things like 'QED' (*quod erat demonstrandum* meaning 'which was to be shown') instead.

Theorem 1.1. The definition of $\mathbf{u}+\mathbf{v}$ does not depend on the point A used to define it. In notation we have $(\mathbf{u}+\mathbf{v})_A = (\mathbf{u}+\mathbf{v})_E$ for all vectors $\mathbf{u}$ and $\mathbf{v}$ and all points A and E.

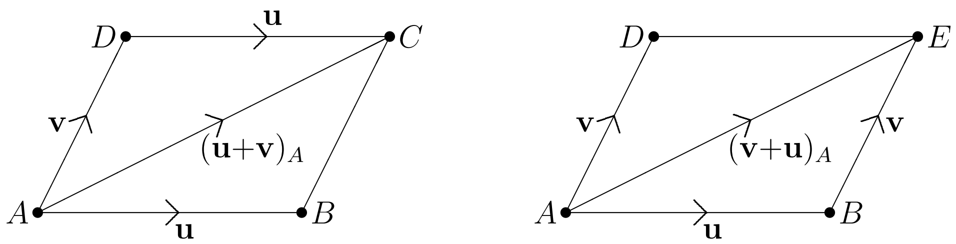
Proof. This proof involves three applications of the Paralellogram Axiom. Let $ABCD$ be the parallelogram obtained by using the Parallelogram Rule for vector addition to calculate $(\mathbf{u}+\mathbf{v})_A$, and let $EFGH$ be the parallelogram obtained by using the Parallelogram Rule for vector addition to calculate $(\mathbf{u}+\mathbf{v})_E$. Thus $\overrightarrow{AB}$, $\overrightarrow{DC}$, $\overrightarrow{EF}$ and $\overrightarrow{HG}$ all represent $\mathbf{u}$, while $\overrightarrow{AD}$ and $\overrightarrow{EH}$ both represent $\mathbf{v}$. Also $\overrightarrow{AC}$ represents $(\mathbf{u}+\mathbf{v})_A$ and $\overrightarrow{EG}$ represents $(\mathbf{u}+\mathbf{v})_E$. Finally, we define $\mathbf{w}$ to be the vector represented by $\overrightarrow{AE}$. All this information is in the diagram below.



Firstly, we examine the quadrilateral $AEHD$, and because two sides, $\overrightarrow{AD}$ and $\overrightarrow{EH}$ represent the same vector (namely $\mathbf{v}$), we conclude that the other two sides $\overrightarrow{AE}$ and $\overrightarrow{DH}$ represent the same vector, which is $\mathbf{w}$. We now turn our attention to the quadrilateral $DHGC$, and note that since $\overrightarrow{DC}$ and $\overrightarrow{HG}$ represent the same vector (namely $\mathbf{u}$), then so do $\overrightarrow{DH}$ and $\overrightarrow{CG}$, this common vector being $\mathbf{w}$. We have now shown that the sides $\overrightarrow{AE}$ and $\overrightarrow{CG}$ of the quadrilateral $AEGC$ represent $\mathbf{w}$, and so applying the Parallelgram Axiom for a third time, we see that $\overrightarrow{AC}$ and $\overrightarrow{EG}$ represent the same vector: that is, we have now shown that $(\mathbf{u}+\mathbf{v})_A = (\mathbf{u}+\mathbf{v})_E$. □

Theorem 1.2. For all vectors $\mathbf{u}$ and $\mathbf{v}$ we have $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$. This law is known as the *commutativity* of vector addition, and we say that vector addition is *commutative*. In fact, $(\mathbf{u}+\mathbf{v})_A = (\mathbf{v}+\mathbf{u})_A$ for all vectors $\mathbf{u}$ and $\mathbf{v}$ and all points A.

Proof. The Parallelogram Rule for vector addition gives us the following parallelograms $ABCD$ and $ABED$ in which $\overrightarrow{AB}$ and $\overrightarrow{DC}$ represent $\mathbf{u}$; $\overrightarrow{AD}$ and $\overrightarrow{BE}$ represent $\mathbf{v}$; $\overrightarrow{AC}$ represents $(\mathbf{u}+\mathbf{v})_A$ and $\overrightarrow{AE}$ represents $(\mathbf{v}+\mathbf{u})_A$ (see the following diagram).

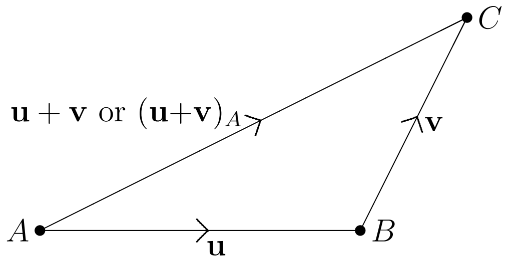


By the Parallelogram Axiom, $\overrightarrow{BC}$ represents $\mathbf{v}$, and by the uniqueness of a point X such that $\overrightarrow{BX}$ represents $\mathbf{v}$ we have that $C = E$. So now both $(\mathbf{u}+\mathbf{v})_A$ and $(\mathbf{v}+\mathbf{u})_A$ are represented by $\overrightarrow{AC}[= \overrightarrow{AE}]$ and therefore $(\mathbf{u}+\mathbf{v})_A = (\mathbf{v}+\mathbf{u})_A$. □

<u>*What is X?*</u> In maths, when one wants to refer to a quantity (so that we can describe some property satisfied by that quantity), we usually have its name, which is typically a letter of the alphabet, such as X or Y. It might be that there is no thing satisfying the properties required by X. For example, there is no real number X such that $X^2 + 1 = 0$. Or X need not be unique; for example there are precisely 3 real numbers X such that $X^3 + X^2 - 2X - 1 = 0$. [It does not matter what the actual values of X are, though in this case I can express them in other terms—one possible value of X is $2\cos\frac{2\pi}{7}$.]

The following gives an alternative way of defining vector addition, known as the *Triangle Rule.*

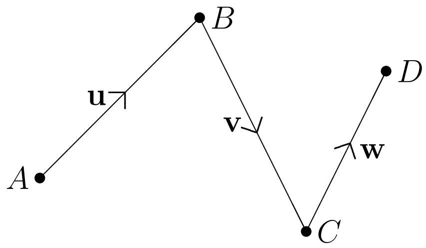
Theorem 1.3 (Triangle Rule)**.** Let A be a point, and let B and C be the unique points such that $\overrightarrow{AB}$ represents $\mathbf{u}$ and $\overrightarrow{BC}$ represents $\mathbf{v}$. Then $\overrightarrow{AC}$ represents $\mathbf{u}+\mathbf{v}$, or more accurately $(\mathbf{u}+\mathbf{v})_A$. The following diagram illustrates the Triangle Rule.



Proof. (For this proof the points A, B, C correspond to the points A, B, E (in that order) in the right-hand parallelogram of the picture in the proof of Theorem 1.2.) By the Parallelogram Rule we see that $\overrightarrow{AC}$ represents $(\mathbf{v}+\mathbf{u})_A$, and by the previous theorem, we have $(\mathbf{v}+\mathbf{u})_A = (\mathbf{u}+\mathbf{v})_A$, hence the result. □

Theorem 1.4. For all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ we have $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$. This property is called *associativity.*

Proof. Pick a point A, and let B, C and D be the unique points such that $\overrightarrow{AB}$ represents $\mathbf{u}$, $\overrightarrow{BC}$ represents $\mathbf{v}$, and $\overrightarrow{CD}$ represents $\mathbf{w}$, see the diagram below.



By the Triangle Rule applied to triangle BCD, we find that $\overrightarrow{BD}$ represents $\mathbf{v}+\mathbf{w}$, and so by the Triangle Rule applied to triangle ABD we obtain that $\overrightarrow{AD}$ represents $\mathbf{u}+(\mathbf{v}+\mathbf{w})$. But the Triangle Rule applied to triangle ABC gives that $\overrightarrow{AC}$ represents $\mathbf{u}+\mathbf{v}$, and applying the Triangle Rule to triangle ACD shows that $\overrightarrow{AD}$ represents $(\mathbf{u}+\mathbf{v})+\mathbf{w}$. Since $\overrightarrow{AD}$ represents both $\mathbf{u}+(\mathbf{v}+\mathbf{w})$ and $(\mathbf{u}+\mathbf{v})+\mathbf{w}$, we conclude that they are equal. □

Theorem 1.5. For all vectors $\mathbf{u}$ we have $\mathbf{u}+\mathbf{0}=\mathbf{u}$. Thus $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$. This asserts that $\mathbf{0}$ is an *identity* for vector addition.

Proof. Exercise (on exercise sheet). □

1.9 Vector subtraction

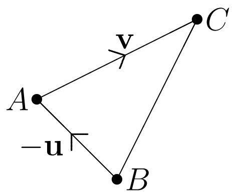
Definition. For vectors $\mathbf{u}$ and $\mathbf{v}$, we define $\mathbf{u}-\mathbf{v}$ by $\mathbf{u}-\mathbf{v} := \mathbf{u}+(-\mathbf{v})$.

Theorem 1.6. For all vectors $\mathbf{u}$ we have $\mathbf{u}-\mathbf{u}=\mathbf{0}$. In other words, for each vector $\mathbf{u}$ we have $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$, and thus $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$ by Theorem 1.2. This property means that $-\mathbf{u}$ is an *additive inverse* of $\mathbf{u}$.

Proof. Let $\overrightarrow{AB}$ represent $\mathbf{u}$. Then $\overrightarrow{BA}$ represents $-\mathbf{u}$, and thus, by the Triangle Rule, $\overrightarrow{AA}$ represents $\mathbf{u}+(-\mathbf{u})$. But $\overrightarrow{AA}$ (also) represents $\mathbf{0}$, and so $\mathbf{u}-\mathbf{u}=\mathbf{u}+(-\mathbf{u})=\mathbf{0}$, as required. □

Theorem 1.7. Suppose $\overrightarrow{AB}$ represents $\mathbf{u}$ and $\overrightarrow{AC}$ represents $\mathbf{v}$. Then $\overrightarrow{BC}$ represents $\mathbf{v}-\mathbf{u}$.

Proof. A diagram for this is as follows.



We have that $\overrightarrow{BA}$ represents $-\mathbf{u}$, and so, by the Triangle Rule, $\overrightarrow{BC}$ represents $(-\mathbf{u})+\mathbf{v}$. But $(-\mathbf{u})+\mathbf{v}=\mathbf{v}+(-\mathbf{u})$ by Theorem 1.2 (commutativity of vector addition), and $\mathbf{v}+(-\mathbf{u})=\mathbf{v}-\mathbf{u}$ by definition. □

1.10 Scalar multiplication

We now define how to multiply a real number α (a *scalar*) by a vector $\mathbf{v}$ to obtain another vector $\alpha\mathbf{v}$. We first specify that $\alpha\mathbf{v}$ has length

$$|\alpha||\mathbf{v}|,$$

where $|\alpha|$ is the absolute value of α and $|\mathbf{v}|$ is the length of $\mathbf{v}$. Thus if $\alpha = 0$ or $\mathbf{v} = \mathbf{0}$ then $|\alpha\mathbf{v}| = |\alpha||\mathbf{v}| = 0$, and so $\alpha\mathbf{v} = \mathbf{0}$. Otherwise (when $\alpha \neq 0$ and $\mathbf{v} \neq \mathbf{0}$), we have that $\alpha\mathbf{v}$ is nonzero, and we must specify the direction of $\alpha\mathbf{v}$. When $\alpha > 0$, we specify that $\alpha\mathbf{v}$ has the same direction as $\mathbf{v}$, and when $\alpha < 0$, we specify that $\alpha\mathbf{v}$ has the same direction as $-\mathbf{v}$ (and hence the opposite direction to $\mathbf{v}$).

Note. I have noticed that some students are writing $\mathbf{v}\alpha$ instead of $\alpha\mathbf{v}$. It is ugly, and I have not defined $\mathbf{v}\alpha$, so please do not use it. I may want to use that notation ($\mathbf{v}\alpha$) for something completely different.

From this definition of scalar multiplication, we observe the following elementary properties.

1. $0\mathbf{v} = \mathbf{0}$ for all vectors $\mathbf{v}$,
2. $\alpha\mathbf{0} = \mathbf{0}$ for all scalars α,
3. $1\mathbf{v} = \mathbf{v}$ for all vectors $\mathbf{v}$,
4. $(-1)\mathbf{v} = -\mathbf{v}$ for all vectors $\mathbf{v}$.

Further properties of scalar multiplication are given below as theorems. These are all harder to prove.

Theorem 1.8. For all vectors $\mathbf{v}$ and all scalars α, β, we have $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$.

Proof. We have $|\alpha(\beta\mathbf{v})| = |\alpha||\beta\mathbf{v}| = |\alpha|(|\beta||\mathbf{v}|) = (|\alpha||\beta|)|\mathbf{v}| = (|\alpha\beta|)|\mathbf{v}| = |(\alpha\beta)\mathbf{v}|$, and so $\alpha(\beta\mathbf{v})$ and $(\alpha\beta)\mathbf{v}$ have the same length.

If $\alpha = 0$, $\beta = 0$ or $\mathbf{v} = \mathbf{0}$ then $\alpha(\beta\mathbf{v}) = \mathbf{0} = (\alpha\beta)\mathbf{v}$ (easy exercise), so we now suppose that $\alpha \neq 0$, $\beta \neq 0$ and $\mathbf{v} \neq \mathbf{0}$, and show that $\alpha(\beta\mathbf{v})$ and $(\alpha\beta)\mathbf{v}$ have the same direction. The rest of the proof breaks into four cases, depending on the signs of α and β.

If $\alpha > 0$, $\beta > 0$ then $\beta\mathbf{v}$ and $(\alpha\beta)\mathbf{v}$ both have the same direction as $\mathbf{v}$, and $\alpha(\beta\mathbf{v})$ has the same direction as $\beta\mathbf{v}$, hence as $\mathbf{v}$. So $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$ in this case. If $\alpha < 0$, $\beta < 0$ then $\alpha\beta > 0$ and both $\alpha(\beta\mathbf{v})$ and $(\alpha\beta)\mathbf{v}$ have the same direction as $\mathbf{v}$, though multiplying by either one of α or β reverses direction.

If $\alpha < 0$, $\beta > 0$ or $\alpha > 0$, $\beta < 0$ then $\alpha(\beta\mathbf{v})$ and $(\alpha\beta)\mathbf{v}$ both have the same direction as $-\mathbf{v}$ (can you see why?). [This is an example of a hidden exercise in the text, and you should still try to do it, even though it will not appear on any exercise sheet.]

So we have now shown that $\alpha(\beta\mathbf{v})$ and $(\alpha\beta)\mathbf{v}$ have the same direction in all cases, completing the proof. □

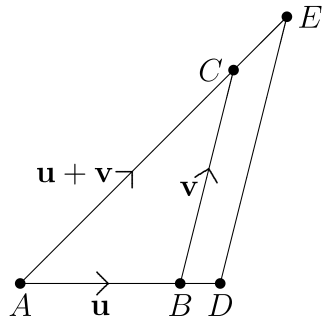
Theorem 1.9. For all vectors $\mathbf{v}$ and all scalars α, β, we have $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$, where the correct bracketing on the right-hand side is $(\alpha\mathbf{v}) + (\beta\mathbf{v})$. This is an example of a *distributive law.*

Proof. Exercise (not on the sheets). When evaluating $\alpha\mathbf{v} + \beta\mathbf{v}$ using the Parallelogram Rule, you may assume that the four (not necessarily distinct) vertices of the parallelogram all lie on a common (straight) line. □

Theorem 1.10. For all vectors $\mathbf{u}$, $\mathbf{v}$ and all scalars α, we have $\alpha(\mathbf{u}+\mathbf{v}) = \alpha\mathbf{u}+\alpha\mathbf{v}$, where the correct bracketing on the right-hand side is $(\alpha\mathbf{u}) + (\alpha\mathbf{v})$. This is also a distributive law.

Proof. This will be at best a sketch of a proof. It will really be an argument convincing you that the result is true using notions from Euclidean geometry. It is better to consider this 'theorem' as an axiom. Our demonstration below will only cover the case $\alpha > 0$, and the diagram is drawn with $\alpha > 1$.

Let $\overrightarrow{AB}$ represent $\mathbf{u}$ and $\overrightarrow{BC}$ represent $\mathbf{v}$, so that $\overrightarrow{AC}$ represents $\mathbf{u}+\mathbf{v}$ by the Triangle Rule. Extending the lines AB and AC as necessary, we let D be the point on the line AB such that $\overrightarrow{AD}$ represents $\alpha\mathbf{u}$, and let E be the point on the line AC such that $\overrightarrow{AE}$ represents $\alpha(\mathbf{u}+\mathbf{v})$.



(case $\alpha > 1$)

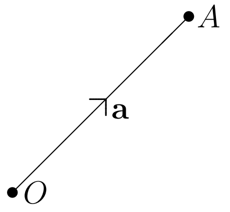
Then $|\overrightarrow{AD}| = \alpha|\overrightarrow{AB}|$ and $|\overrightarrow{AE}| = \alpha|\overrightarrow{AC}|$ and so triangles ABC and ADE are similar (that is one is a scaling of the other; here this scaling fixes the point A). Therefore $|\overrightarrow{DE}| = \alpha|\overrightarrow{BC}|$ and $\overrightarrow{BC}$ and $\overrightarrow{DE}$ have the same direction, and so $\overrightarrow{DE}$ represents $\alpha\mathbf{v}$. Now we use the Triangle Rule with triangle ADE to conclude that $\alpha\mathbf{u}+\alpha\mathbf{v} = \alpha(\mathbf{u}+\mathbf{v})$.

(Note that the notion of similarity in general allows translations, rotations and reflexions as well as scaling.) □

Aside. In 2011, one student observed that Theorem 1.4 (among others) looked more like an axiom than a theorem. It is quite possible to choose an alternative axiomatisation of geometry in which several of the theorems here (including Theorem 1.4) are axioms. It is also very likely that the Parallelogram Axiom would no longer be an axiom in such an alternative system, but a theorem requiring proof.

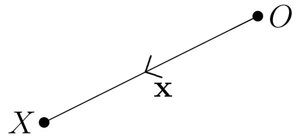
1.11 Position vectors

In order to talk about position vectors, we need to assume that we have fixed a point O as an *origin* in 3-space. Then if A is any point, the *position vector* of A is defined to be the free vector represented by the bound vector $\overrightarrow{OA}$.



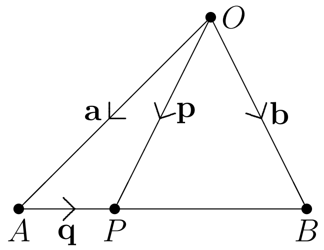
a is the position vector of the point A

Note that each vector $\mathbf{x}$ is the position vector of exactly one point X in 3-space. This point X has as distance and direction from the origin the length and direction of the vector $\mathbf{x}$.



Theorem 1.11. Let A and B be points with position vectors $\mathbf{a}$ and $\mathbf{b}$. Let P be a point on the line segment AB such that $|\overrightarrow{AP}| = \lambda|\overrightarrow{AB}|$. Then P has position vector $\mathbf{p} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$.

Proof. A diagram for this situation is as follows.

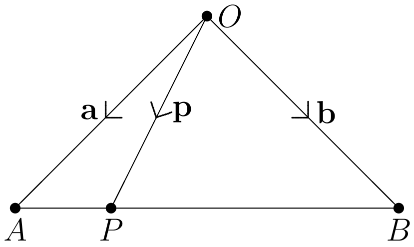


Let $\mathbf{c}$ be the vector represented by $\overrightarrow{AB}$, and let $\mathbf{q}$ be the vector represented by $\overrightarrow{AP}$. Then $\mathbf{q} = \lambda\mathbf{c}$. Now $\mathbf{a} + \mathbf{c} = \mathbf{b}$ (by the Triangle Rule), and adding $-\mathbf{a}$ to both sides we obtain $\mathbf{c} = \mathbf{b} - \mathbf{a}$. Therefore, using the Triangle Rule, we get $\mathbf{p} = \mathbf{a} + \mathbf{q} = \mathbf{a} + \lambda\mathbf{c}$, and using the various rules for vector addition and scalar multiplication, we get:

$$\begin{aligned}
\mathbf{p} = \mathbf{a} + \lambda\mathbf{c} &= \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) \\
&= \mathbf{a} + \lambda(\mathbf{b} + (-\mathbf{a})) \\
&= \mathbf{a} + \lambda\mathbf{b} + \lambda(-\mathbf{a}) \\
&= \mathbf{a} + \lambda\mathbf{b} + \lambda((-1)\mathbf{a}) \\
&= 1\mathbf{a} + \lambda\mathbf{b} + (-\lambda)\mathbf{a} \\
&= (1-\lambda)\mathbf{a} + \lambda\mathbf{b},
\end{aligned}$$

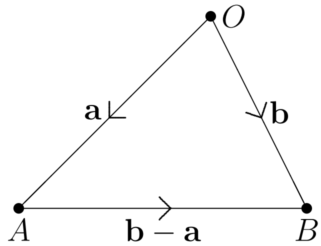
as required. □

Example. Suppose P is one quarter of the way from A along the line segment AB. Then $\mathbf{p} = (1 - \frac{1}{4})\mathbf{a} + \frac{1}{4}\mathbf{b} = \frac{3}{4}\mathbf{a} + \frac{1}{4}\mathbf{b}$ (see picture overleaf).



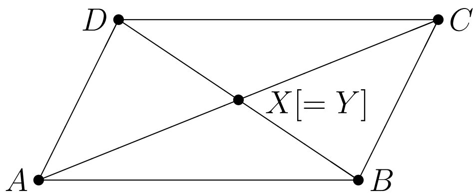
$|\overrightarrow{AP}| = \frac{1}{4}|\overrightarrow{AB}|$

Note. If A and B are any points, with position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, then the vector represented by $\overrightarrow{AB}$ is $\mathbf{b} - \mathbf{a}$ (see picture below).



Theorem 1.12 (Application of Theorem 1.11)**.** The diagonals of a parallelogram $ABCD$ meet each other in their midpoints.

Proof. This proof proceeds by determining the midpoints of the two diagonals and showing they are the same. The diagram of what we wish to prove is given below.



Let X be the midpoint of the diagonal AC, and let Y be the midpoint of the diagonal BD. Let A, B, C, D, X, Y have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{x}$, $\mathbf{y}$ respectively. Then, by Theorem 1.11, we have

$$\mathbf{x} = (1 - \tfrac{1}{2})\mathbf{a} + \tfrac{1}{2}\mathbf{c} = \tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{c} = \tfrac{1}{2}(\mathbf{a} + \mathbf{c}) \quad \text{and} \quad \mathbf{y} = (1 - \tfrac{1}{2})\mathbf{b} + \tfrac{1}{2}\mathbf{d} = \tfrac{1}{2}(\mathbf{b} + \mathbf{d}).$$

Since $ABCD$ is a parallelogram, $\overrightarrow{AB}$ and $\overrightarrow{DC}$ represent the same vector. Since $\overrightarrow{AB}$ represents $\mathbf{b} - \mathbf{a}$ and $\overrightarrow{DC}$ represents $\mathbf{c} - \mathbf{d}$, we have $\mathbf{b} - \mathbf{a} = \mathbf{c} - \mathbf{d}$. Adding $\mathbf{a} + \mathbf{d}$ to both sides (and using the rules of vector addition and subtraction) gives $\mathbf{b} + \mathbf{d} = \mathbf{c} + \mathbf{a} [= \mathbf{a} + \mathbf{c}]$. So now $\frac{1}{2}(\mathbf{a} + \mathbf{c}) = \frac{1}{2}(\mathbf{b} + \mathbf{d})$, which implies that $\mathbf{x} = \mathbf{y}$, whence $X = Y$. □

In lectures, the symbol $\Rightarrow$ often crops up, especially in proofs. It means 'implies' or 'implies that.' So $A \Rightarrow B$ means 'A implies B' or 'if A then B' or 'A only if B.' The symbol $\Leftarrow$ means 'is implied by,' so that $A \Leftarrow B$ means 'A is implied by B' or 'A if B.' The symbol $\Leftrightarrow$ means 'if and only if' so that $A \Leftrightarrow B$ means 'A if and only if B' or 'A is equivalent to B.' Throughout the above, A and B are statements.