MTH4103: Geometry I

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Preface

These are notes for the course MTH4103: Geometry I that I gave (am giving) in Semester B of the 2013–14 academic year at Queen Mary, University of London. These notes are based on handwritten notes I inherited from Professor L. H. Soicher, who lectured this course in the academic years 2005–06 to 2009–10. The notes were typed up (using the LaTeX document preparation system) for the 2010–11 session by the two lecturers that year, namely Dr J. N. Bray (Chapters 1–6) and Prof. S. R. Bullett (Chapters 7–10). Naturally, several modifications were made to the notes in the process of typing them up, as one expects to happen when a new lecturer takes on a course. I have made many further revisions to the notes last year, and a few more this year, including some to take advantage of the new module MTH4110: Mathematical Structures. Since I have now typed up and/or edited the whole set of notes, the *culpa* for any errors, omissions or infelicities therein is entirely *mea*.

Despite the presence of these notes, one should still take (or have taken) notes in lectures. Certainly the examples given in lectures differ from those in these notes, and there is material I covered in the lectures that does not appear in these notes, and *vice versa*. Furthermore, the examinable material for the course is defined by what was covered in lectures (including the proofs). In these notes, I have seldom indicated which material is examinable.

My thanks go out to those colleagues who covered the several lectures I missed owing to illness.

Dr John N. Bray, 25th March 2014