Queen Mary
University of London

B. Sc. Examination by course unit 2013

MTH4103: Geometry I

Duration: 2 hours

Date and time: 3rd May 2013, 14:30–16:30

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Examiner(s): J. N. Bray

Question 1 Let $A = (2, -1, -2)$ and $B = (-3, 2, 5)$, and let $\mathbf{a}$ and $\mathbf{b}$ be the position vectors of A and B respectively. Determine:

(a) the length of $\mathbf{a}$; [2]

(b) the vector having length 2 in the same direction as $\mathbf{a}$; [3]

(c) the vector represented by $\overrightarrow{AB}$; [2]

(d) a vector equation for the line through A and B; [2]

(e) the cosine of the angle between $\mathbf{a}$ and $\mathbf{b}$. [3]

(f) the vector product $\mathbf{a} \times \mathbf{b}$ of $\mathbf{a}$ and $\mathbf{b}$. [4]

Question 2 Let $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$, and let $\mathbf{b}$ and $\mathbf{c}$ be position vectors of points in the (x, y)-plane such that the area of the parallelogram with sides $\mathbf{b}$ and $\mathbf{c}$ is 7. Let the parallelepiped with sides corresponding to $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ have volume V.

(a) Is there sufficient information to calculate V? [Answer: 'Yes' or 'No'.] [2]

(b) (i) If your answer to Part (a) was 'Yes', determine V.

(ii) If your answer to Part (a) was 'No', specify an extra piece (or extra pieces) of information necessary and sufficient to determine V. [If you specify redundant information you will gain no marks.] [2]

Question 3

(a) Use Gaußian elimination (to reduce to echelon form) followed by back substitution to determine *all* solutions to the following system of linear equations in x, y, z defined over $\mathbb{R}$:

$$\left.\begin{aligned} -x + y + 3z &= 12 \\ 2x + y \qquad &= -3 \\ x + y + \ z &= 2 \end{aligned}\right\}.$$

[8]

(b) What exactly does your answer to Part (a) tell you about the intersection of the three planes defined by the equations above? [2]

Question 4 Calculate the distance from the point $B = (1,-1,2)$ to the plane Π defined by the Cartesian equation $2x - y + 4z = 13$. [4]

Question 5 Let

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

(a) Find:

(i) $-2A + 3\mathrm{I}_3$; [3]

(ii) A^2; [3]

(iii) the characteristic polynomial of A; [5]

(iv) $\det A$ [Hint: Evaluate the characteristic polynomial at 0]; [2]

(v) the eigenvalues of A (regarded as a real matrix). [3]

(b) Is A invertible? Justify your answer. [There is no need to evaluate A^{-1} in the event that A be invertible.] [2]

Question 6

(a) Define precisely, **without using coördinates**, the *vector product* $\mathbf{u} \times \mathbf{v}$ of vectors $\mathbf{u}$ and $\mathbf{v}$. [5]

(b) Prove, using your definition in Part (a), that $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$ for all vectors $\mathbf{u}$ and $\mathbf{v}$. [7]

Question 7 Let the points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. Consider the triangle ABC, in which the mid-points of the edges BC, CA, AB are P, Q, R respectively. Show that the lines AP, BQ and CR meet at the point X with position vector $\mathbf{x} = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$. [You may use the notation that P, Q, R have position vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$, without stating this explicitly in your solution.] [6]

Question 8

(a) Define what it means for a map $t : \mathbb{R}^m \to \mathbb{R}^n$ to be a *linear transformation*. [4]

(b) Let $t : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Prove that $t(\mathbf{0}_m) = \mathbf{0}_n$, and that $t(-\mathbf{v}) = -t(\mathbf{v})$ for all $\mathbf{v} \in \mathbb{R}^m$. [6]

(c) Define the map $t : \mathbb{R}^1 \to \mathbb{R}^1$ by $t(\begin{pmatrix} x \end{pmatrix}) = \begin{pmatrix} x^3 \end{pmatrix}$ for all $x \in \mathbb{R}$. Is t a linear transformation? Justify your answer. [4]

Question 9

(a) Define what it means to say, **without using determinants**, that an $n \times n$ matrix A is *invertible*, and what is meant by the *inverse* of A (when A is invertible). [4]

(b) Define what is meant by an *eigenvector* of an $n \times n$ matrix A, and the *eigenvalue* corresponding to that eigenvector. [4]

(c) Suppose that A and B are $n \times n$ matrices such that $A^2 = B^2 = (AB)^2 = \mathrm{I}_n$. Prove that $AB = BA$. [4]

(d) Suppose that A is an $n \times n$ matrix such that $A^2 = \mathrm{I}_n$. Determine the possible eigenvalues of A. [These might not all occur for a particular matrix A.] [4]

End of Paper