Queen Mary
University of London

B. Sc. Examination by course unit 2012

MTH4103: Geometry I

Duration: 2 hours

Date and time: 4th May 2012, 10:00–12:00

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You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Examiner(s): J. N. Bray

Question 1 Let $A = (1,-1,2)$ and $B = (2,1,7)$, and let $\mathbf{a}$ and $\mathbf{b}$ be the position vectors of A and B respectively. Determine:

(a) the length of $\mathbf{a}$; [2]

(b) the vector having length 6 and *opposite* direction to $\mathbf{a}$; [3]

(c) the vector represented by $\overrightarrow{AB}$; [2]

(d) a vector equation for the line through A and B; [2]

(e) the cosine of the angle between $\mathbf{a}$ and $\mathbf{b}$. [3]

Question 2 Let $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$. Determine:

(a) the area of a parallelogram with sides corresponding to $\mathbf{v}$ and $\mathbf{w}$; and [5]

(b) the volume of a parallelepiped with sides corresponding to $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. [3]

Question 3

(a) Use Gaußian elimination (to reduce to echelon form) followed by back substitution to determine *all* solutions to the following system of linear equations in x, y, z defined over $\mathbb{R}$:

$$\left.\begin{array}{r} x+y+\ \ z=2 \\ 2x-y+3z=3 \\ 4x+y+9z=7 \end{array}\right\}.$$

[6]

(b) What exactly does your answer to Part (a) tell you about the intersection of the three planes defined by the equations above? [2]

Question 4 Calculate the distance from the point $C = (1,-1,2)$ to the line ℓ defined by the Cartesian equations

$$\frac{x+1}{2} = \frac{y-2}{3} = \frac{z-3}{-1}.$$

[6]

Question 5 Let

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 1 & -3 & -2 \\ -1 & 1 & 1 \end{pmatrix}.$$

Find:

(a) $-A + 3\mathrm{I}_3$; [4]

(b) A^2; [4]

(c) $\det A$. [4]

Question 6

(a) Define what it means to say, **without using determinants**, that an $n \times n$ matrix A is *invertible*, and what is meant by the *inverse* of A. [4]

(b) Let S_θ denote the 2×2 matrix representing the reflexion (in the (x, y)-plane) in the line through the origin at anticlockwise angle $\theta/2$ to the x-axis. Then

$$S_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

(i) Prove that S_θ is invertible, and calculate S_θ^{-1}. [4]

(ii) Determine the eigenvalues of S_θ. [4]

Question 7

(a) Define precisely, **without using coördinates**, the *scalar product* $\mathbf{u}\cdot\mathbf{v}$ of vectors $\mathbf{u}$ and $\mathbf{v}$. [4]

(b) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors. Define precisely what it means for $\mathbf{u}$ to be *parallel* to $\mathbf{v}$. [4]

(c) Prove that $|\mathbf{u}\cdot\mathbf{v}| \leqslant |\mathbf{u}||\mathbf{v}|$ for all vectors $\mathbf{u}$ and $\mathbf{v}$. Determine precisely when equality holds. [8]

Question 8

(a) Define what it means for a map $t : \mathbb{R}^m \to \mathbb{R}^n$ to be a *linear transformation*. [4]

(b) For $\mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$, define the map $r_\mathbf{x} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$r_\mathbf{x}(\mathbf{r}) = \mathbf{r} - 2\left(\frac{\mathbf{r}\cdot\mathbf{x}}{\mathbf{x}\cdot\mathbf{x}}\right)\mathbf{x}.$$

Prove that $r_\mathbf{x}$ is a linear transformation. [6]

(c) The map $t : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $t(\mathbf{r}) = (\mathbf{i}\cdot\mathbf{r})\mathbf{r}$. Determine, with justification, whether t is a linear transformation.
[In this question $\mathbf{i}$ is one of the standard unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.] [4]

Question 9

(a) Define what is meant by an *eigenvector* of an $n \times n$ matrix A, and the *eigenvalue* corresponding to that eigenvector. [4]

(b) Suppose that the $n \times n$ matrix A has λ as an eigenvalue. Prove that A^2 has λ^2 as an eigenvalue. [4]

(c) Suppose that A has eigenvectors $\mathbf{u}$ and $\mathbf{v}$ with corresponding eigenvalues λ and μ, where $\lambda \neq \mu$. Prove that $\mathbf{u}+\mathbf{v}$ is *not* an eigenvector of A.
[You may assume that $\mathbf{u}$ and $\mathbf{v}$ are *not* collinear, that is, if α and β are scalars such that $\alpha\mathbf{u}+\beta\mathbf{v} = \mathbf{0}$ then $\alpha = \beta = 0$.] [4]

End of Paper