Queen Mary
University of London

B. Sc. Examination by course unit 2011

MTH4103 Geometry I

Duration: 2 hours

Date and time: 18th May 2011, 14:30–16:30

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorised materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): J. N. Bray and S. R. Bullett

Question 1 Let $A = (1,2,3)$ and $B = (2,-1,-1)$, and let $\mathbf{a}$ and $\mathbf{b}$ be the position vectors of A and B respectively. Determine:

(a) the length of $\mathbf{a}$; [2]

(b) the vector represented by $\overrightarrow{AB}$; [2]

(c) parametric equations for the line through A and B; [3]

(d) the cosine of the angle between $\mathbf{a}$ and $\mathbf{b}$; [3]

(e) the vector product $\mathbf{a} \times \mathbf{b}$ of $\mathbf{a}$ and $\mathbf{b}$. [4]

Question 2

(a) By using Gaußian elimination to reduce to echelon form, followed by back substitution, find *all* solutions to the following system of linear equations in x, y, z defined over $\mathbb{R}$:

$$\left.\begin{aligned} 4z &= 8 \\ x - 2y + 2z &= 0 \\ -2x + 4y + 5z &= 18 \end{aligned}\right\} .$$

[8]

(b) What exactly does your answer to Part (a) tell you about the intersection of the three planes defined by the equations above? [2]

Question 3 Calculate the distance from the point $C = (1,-1,4)$ to the plane Π defined by the equation $2x - y + 3z = 4$. [4]

Question 4 Let

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 7 & -5 \\ 0 & 8 & -6 \end{pmatrix} .$$

Find:

(a) $\det(A)$; [4]

(b) the characteristic polynomial of A; [4]

(c) all eigenvalues of A. [4]

Question 5

(a) Define what it means to say, **without using determinants**, that an $n \times n$ matrix A is *invertible*, and what is meant by the *inverse* of A. [4]

(b) Let

$$A = \begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}.$$

Determine whether A is invertible and, if it is, find A^{-1}. [4]

(c) Suppose that A and B are invertible $n \times n$ matrices. Prove that their product AB is invertible, and show that $(AB)^{-1} = B^{-1}A^{-1}$. [4]

Question 6

(a) For each of the following linear transformations of $\mathbb{R}^2$, determine the 2×2 matrix which represents it:

(i) the reflection in the line $y = -x$; [2]

(ii) the rotation about the origin through an (anticlockwise) angle of $\pi/2$; [2]

(iii) the transformation t obtained by first performing a reflection in the line $y = -x$ followed by a rotation about the origin through an (anticlockwise) angle of $\pi/2$. [3]

(b) Is the transformation t constructed in (a) (iii) above a reflection or a rotation? If it is a reflection, find an equation for the line it fixes pointwise (the mirror); if it is a rotation, find the angle through which it turns. [3]

Question 7 (a) Suppose vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$ satisfy the properties that $\mathbf{a} \cdot \mathbf{u} = 0$ for all $\mathbf{u} \in \mathbb{R}^3$ and $\mathbf{b} \times \mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in \mathbb{R}^3$. Prove that $\mathbf{a} = \mathbf{b} = \mathbf{0}$. [6]

(b) Consider the map $t : \mathbb{R}^3 \to \mathbb{R}^3$ given by $t(\mathbf{r}) = \mathbf{r} + \mathbf{k}$ for $\mathbf{r} \in \mathbb{R}^3$, where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ form the standard right-handed triple of pairwise orthogonal unit vectors.

(i) Write down the map t when $\mathbf{r}$ and its image under t are written in terms of coördinates. [2]

(ii) Is t a linear transformation? Justify your answer. [2]

Question 8

(a) Define precisely, **without using coördinates**, the *vector product* $\mathbf{u} \times \mathbf{v}$ of vectors $\mathbf{u}$ and $\mathbf{v}$. [Take good care of the degenerate cases.] [5]

For the remainder of this question, standard results about the vector product may be assumed without proof. Let $KLMN$ be a parallelogram, and let the sides $\overrightarrow{KL}$ and $\overrightarrow{KN}$ represent the vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, and let the diagonals $\overrightarrow{KM}$ and $\overrightarrow{LN}$ represent the vectors $\mathbf{c}$ and $\mathbf{d}$ respectively.

(b) Write down the area A of $KLMN$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

(c) Write down $\mathbf{c}$ and $\mathbf{d}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. Thus write down $\mathbf{a}$ and $\mathbf{b}$ in terms of $\mathbf{c}$ and $\mathbf{d}$ and express the area A in terms of $\mathbf{c}$ and $\mathbf{d}$. [6]

(d) Write $\mathbf{b}$ in terms of $\mathbf{a}$ and $\mathbf{c}$, and thus express the area A in terms of $\mathbf{a}$ and $\mathbf{c}$. [3]

Question 9

(a) Define what is meant by an *eigenvector* of an $n \times n$ matrix A, and the *eigenvalue* corresponding to that eigenvector. [4]

(b) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of A with the same eigenvalue λ, and that $\mathbf{u} + \mathbf{v} \neq \mathbf{0}$. Show that $\mathbf{u} + \mathbf{v}$ is an eigenvector of A. [4]

(c) Suppose that $\mathbf{u}$ is an eigenvector of A and that B is an $n \times n$ matrix with $AB = BA$ and $B\mathbf{u} \neq \mathbf{0}$. Show that $B\mathbf{u}$ is also an eigenvector of A. [4]

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