B.Sc. Examination by Course Unit

MAS114 Geometry I

9 May 2008, 10:00–12:00

The duration of this examination is 2 hours.
You should attempt all questions. Marks awarded are shown next to the questions.
Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Do NOT start reading the question paper until instructed to do so by the invigilator.

1. Let $\underline{u} = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$, $\underline{v} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$, and $\underline{w} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$. Calculate the following:

(i) [2 marks] $|\underline{u}|$;

(ii) [2 marks] the vector of length 1 having the same direction as $\underline{u}$;

(iii) [2 marks] $(\underline{u} \cdot \underline{v})\underline{w}$;

(iv) [2 marks] $\underline{v} \times \underline{w}$;

(v) [4 marks] the volume of a parallelepiped with sides corresponding to $\underline{u}, \underline{v}, \underline{w}$.

2. Let $A = (1, 2, 3)$ and $B = (2, 5, 7)$. Determine:

(i) [2 marks] the vector represented by $\overrightarrow{AB}$;

(ii) [4 marks] Cartesian equations for the line through the points A and B;

(iii) [6 marks] a Cartesian equation for the plane through the points A, B, and $(-1, 1, 3)$.

3. (i) [6 marks] Determine all solutions of the following system of linear equations in x, y, z:

$$\begin{cases} x + 2y + z = 2 \\ 3x + 5y + 2z = 6 \\ -2x - 2y + z = 1 \end{cases}.$$

(ii) [2 marks] What exactly does your answer to part (i) tell you about the intersection of the three planes defined by the equations above?

4. [6 marks] Calculate the distance from the point $(-1, 2, 3)$ to the line defined by the vector equation

$$\underline{r} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

Next question overleaf

5. Let $A = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}$. Determine:

(i) [2 marks] $2A - 3I_2$;

(ii) [2 marks] A^2;

(iii) [2 marks] $\det(A)$;

(iv) [2 marks] whether A is invertible, and if so, A^{-1};

(v) [2 marks] the characteristic polynomial of A;

(vi) [2 marks] all the eigenvalues of A.

6. (i) [4 marks] Define what it means for a function $t : \mathbb{R}^n \to \mathbb{R}^m$ to be a *linear transformation*.

(ii) [6 marks] For each of the following linear transformations of $\mathbb{R}^2$, determine the 2×2 matrix representing that transformation:

(a) a reflection in the x-axis;

(b) a rotation about the origin through a counterclockwise angle of $\pi/2$;

(c) the transformation made by performing a reflection in the x-axis followed by a rotation about the origin through a counterclockwise angle of $\pi/2$.

7. [4 marks] For each of the following statements, say whether it is true or false. Your answer must be correct in all cases to obtain marks.

(i) $\det(A + B) = \det(A) + \det(B)$ for all 3×3 matrices A and B.

(ii) $\det(\alpha A) = \alpha \det(A)$, for all scalars α and all 3×3 matrices A.

(iii) $\det(-A) = -\det(A)$, for all 3×3 matrices A.

(iv) If A is an invertible 2×2 matrix, then we must have $\det(A) \neq \det(A^{-1})$.

Next question overleaf

8. (i) [5 marks] Define precisely, not using co-ordinates, the *vector product* $\underline{u} \times \underline{v}$ of vectors $\underline{u}$ and $\underline{v}$.

(ii) [9 marks] Apply your definition in part (i) to prove, without using co-ordinates, that

$$\underline{v} \times \underline{u} = -(\underline{u} \times \underline{v}).$$

9. [9 marks] Let $A = (a_{ij})_{m\times n}$, $X = (x_{ij})_{n\times p}$, and let α be a scalar. Prove that

$$(\alpha A)X = \alpha(AX).$$

10. (i) [4 marks] Define what is meant by an *eigenvalue* of an $n \times n$ matrix A.

(ii) [9 marks] Let A be an $n \times n$ matrix (with $n > 0$) and let λ be a scalar, such that, for each row of A, the sum of the entries in that row is equal to λ.

Prove that λ is an eigenvalue of A.

End of question paper