Queen Mary, University of London
B.Sc. Examination

MAS114 Geometry I

17 May 2007, 14:30–16:30

The duration of this examination is 2 hours.
You should attempt all questions. Marks awarded are shown next to the questions.
Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Do NOT start reading the question paper until instructed to do so by the invigilator.

1. Let $\underline{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$. Calculate the following:

(i) [2 marks] $2\underline{u} - 3\underline{v}$;

(ii) [2 marks] a vector of length 1 in the same direction as $\underline{u}$;

(iii) [2 marks] a vector equation for the line through $(1, 3, -4)$ and in the same direction as $\underline{u}$.

2. Let $\underline{u} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$. Calculate the following:

(i) [2 marks] $\underline{u} \times \underline{v}$;

(ii) [2 marks] a non-zero vector $\underline{w}$ such that $\underline{w}$ is orthogonal to both $\underline{u}$ and $\underline{v}$, and $\underline{u}, \underline{v}, \underline{w}$ is a left-handed triple;

(iii) [3 marks] a Cartesian equation for the plane through the point $(-1, 4, 5)$ and parallel to $\underline{u}$ and $\underline{v}$;

(iv) [4 marks] the distance of the point $(1, 2, 3)$ from the plane defined by $x + 2y - 4z = 5$.

3. (i) [6 marks] By reducing to echelon form, determine all solutions to the following system of linear equations in x, y, z:

$$\begin{cases} 2x + 2y + 2z = 0 \\ \quad\;\; 2y + 4z = 0 \\ x + 2y + 3z = 0 \end{cases}.$$

(ii) [2 marks] What exactly does your answer to (i) tell you about the intersection of the three planes defined by the equations above?

4. (i) [4 marks] Calculate $\det(A)$, where $A = \begin{pmatrix} -1 & 1 & -2 \\ 2 & 2 & 3 \\ 1 & -2 & 3 \end{pmatrix}$.

(ii) [4 marks] Find the volume of the parallelepiped with sides corresponding to $\underline{u}, \underline{v}, \underline{w}$, where $\underline{u} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, $\underline{v} = \begin{pmatrix} 3 \\ -6 \\ -3 \end{pmatrix}$, $\underline{w} = \begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix}$.

Are $\underline{u}, \underline{v}, \underline{w}$ coplanar? If $\underline{u}, \underline{v}, \underline{w}$ are not coplanar then are they a right-handed triple or a left-handed triple? Justify your answers.

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5. [6 marks] Consider the parallelogram $ABCD$, with $A = (1, 2, 1)$, $B = (1, -1, 3)$, and $C = (2, 1, 0)$. Determine the area of $ABCD$.

6. For the matrix $A = \begin{pmatrix} -1 & 1 \\ 3 & 1 \end{pmatrix}$, do the following:

(i) [2 marks] calculate $A^2 + 3I_2$;

(ii) [3 marks] determine if A is invertible, and if so, find A^{-1};

(iii) [2 marks] determine the characteristic polynomial of A;

(iv) [5 marks] find all the eigenvalues of A, and for each such eigenvalue λ, find the set of all eigenvectors of A with corresponding eigenvalue λ.

7. (i) [2 marks] Define what it means for a function $t : \mathbb{R}^n \to \mathbb{R}^m$ to be a *linear transformation.*

(ii) [4 marks] Let $s : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation with

$$s\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \; s\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix},$$

and let $t : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation with

$$t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}, \; t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \; t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Determine the matrices representing s, t, and $s \circ t$.

8. (i) [2 marks] With respect to a fixed origin O in 3-dimensional space, what is meant by the *position vector* $\underline{a}$ of a point A?

(ii) [6 marks] Let A and B be points with respective position vectors $\underline{a}$ and $\underline{b}$, and let P be a point on the line segment AB such that $|\overrightarrow{AP}| = \lambda|\overrightarrow{AB}|$. Prove that P has position vector

$$\underline{p} = (1 - \lambda)\underline{a} + \lambda\underline{b}.$$

(iii) [6 marks] Let A, B, C, D be any four points in 3-dimensional space, and let P, Q, R, S be the respective mid-points of the line segments AB, BC, CD, DA. Prove that $PQRS$ is a parallelogram.

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9. (i) [3 marks] Define (not using co-ordinates) the *scalar product* $\underline{u} \cdot \underline{v}$ of two vectors $\underline{u}$ and $\underline{v}$.

(ii) [8 marks] Suppose vectors $\underline{u}$,$\underline{v}$ are given in coordinates by $\underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$.

(a) State a formula (without proof) for $\underline{u} \cdot \underline{v}$ in terms of the co-ordinates of $\underline{u}$ and $\underline{v}$.

(b) Assuming the formula in (a), prove that $\underline{u} \cdot (\alpha\underline{v} + \underline{w}) = \alpha(\underline{u} \cdot \underline{v}) + \underline{u} \cdot \underline{w}$, for all vectors $\underline{u}, \underline{v}, \underline{w}$ and scalars α.

(iii) [4 marks] Exactly which of the following statements are true? [Your answer must be completely correct to obtain marks.]

(a) If $\underline{u}$ and $\underline{v}$ are vectors such that $\underline{u} \cdot \underline{v} = 0$, then we must have $\underline{u} = \underline{0}$ or $\underline{v} = \underline{0}$.

(b) If $\underline{u}$ is a vector such that $\underline{u} \cdot \underline{v} = 0$ for every vector $\underline{v}$, then we must have $\underline{u} = \underline{0}$.

(c) $\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$ for all vectors $\underline{u}, \underline{v}$.

(d) If $\underline{u}$ and $\underline{v}$ are non-zero vectors such that $|\underline{u} \cdot \underline{v}| = |\underline{u} \times \underline{v}|$ then $\underline{u}$ and $\underline{v}$ must be parallel.

(e) If $\underline{u}$ and $\underline{v}$ are non-zero vectors such that $|\underline{u} \cdot \underline{v}| = |\underline{u} \times \underline{v}|$ then $\underline{u}$ and $\underline{v}$ must be orthogonal.

10. (i) [2 marks] Write down the 2×2 matrix representing the rotation r_θ in the (x, y)-plane through a counterclockwise angle θ about the origin.

(ii) [6 marks] Prove that if θ is not an integer multiple of π, then r_θ has no (real) eigenvalue.

(iii) [6 marks] Write down the 3×3 matrix representing the reflection (in 3 dimensions) in the plane Π which goes through the origin and is orthogonal to $\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$.

End of question paper