MTH4103 (2013–14)

Geometry I

Solutions 10

26th March 2014

Practice Question 1. This definition is in the Week 12 lecture notes. Let t be a linear transformation of $\mathbb{R}^n$ represented by the $n \times n$ matrix A.
We call $\mathbf{v} \in \mathbb{R}^n$ an *eigenvector* of t (and of A) if $\mathbf{v} \neq \mathbf{0}_n$ and $t(\mathbf{v}) = A\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ, in which case λ is called the *eigenvalue* of t (and of A) corresponding to $\mathbf{v}$.

Practice Question 2. Note that the zero vector should never be included as an eigenvector.

(a) Let $A = \begin{pmatrix} 1 & -1 \\ 3 & 5 \end{pmatrix}$. The characteristic polynomial of A is $\det(A - x\mathrm{I}_2) = \begin{vmatrix} 1-x & -1 \\ 3 & 5-x \end{vmatrix} = (1-x)(5-x) - (-3) = x^2 - 6x + 8 = (x-2)(x-4)$. Thus the eigenvalues of A are 2 and 4.

To determine the eigenvectors with corresponding eigenvalue 2, we solve

$$(A - 2\mathrm{I}_2)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{that is} \quad \begin{pmatrix} -1 & -1 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is equivalent to the single equation $-x - y = 0$ (the other equation is $3x + 3y = 0$, which is a scalar multiple of this one). Thus x can be any real number, r say, and then $y = -r$. (Using Gaußian elimination strictly we would set $y = t$, where t can be any real number, and conclude that $x = -t$.) Thus, the set of all eigenvectors with corresponding eigenvalue 2 is

$$\left\{ \begin{pmatrix} r \\ -r \end{pmatrix} : r \in \mathbb{R} \;\middle|\; r \neq 0 \right\}.$$

To determine the eigenvectors with corresponding eigenvalue 4, we solve

$$(A - 4\mathrm{I}_2)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{that is} \quad \begin{pmatrix} -3 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is equivalent to the single equation $-3x - y = 0$ (the other equation is $3x + y = 0$, which is a scalar multiple of this one). Thus x can be any real number, r say, and then $y = -3r$. (Using Gaußian elimination strictly we would set $y = t$, where t can be any real number, and conclude that $x = -t/3$.) Thus, the set of all eigenvectors with corresponding eigenvalue 4 is

$$\left\{ \begin{pmatrix} r \\ -3r \end{pmatrix} : r \in \mathbb{R} \;\middle|\; r \neq 0 \right\}.$$

(b) Let $A = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$. The characteristic polynomial of A is $\det(A - x\mathrm{I}_2) = \begin{vmatrix} 4-x & 0 \\ 0 & 4-x \end{vmatrix} = (4-x)^2 = (x-4)^2$. Thus the only eigenvalue of A is 4.

To determine the eigenvectors with corresponding eigenvalue 4, we solve

$$(A - 4\mathrm{I}_2)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{that is} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and observe that **all** $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ are solutions.

[Alternatively, observe that $A\mathbf{v} = 4\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^2$.]

Thus, the set of all eigenvectors with corresponding eigenvalue 4 consists of all the elements of $\mathbb{R}^2$ except for the zero vector, that is, $\mathbb{R}^2 \setminus \{\mathbf{0}_2\}$, or equivalently

$$\left\{ \mathbf{v} \in \mathbb{R}^2 \;\middle|\; \mathbf{v} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

(c) Let $A = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix}$. The characteristic polynomial of A is $\det(A - x\mathrm{I}_2) = \begin{vmatrix} -2-x & -1 \\ 1 & -4-x \end{vmatrix} = (-2-x)(-4-x) - (-1) = x^2 + 6x + 9 = (x+3)^2$. Thus the only eigenvalue of A is -3.

To determine the eigenvectors with corresponding eigenvalue -3, we solve

$$(A + 3\mathrm{I}_2)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{that is} \quad \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is equivalent to the single equation $x - y = 0$ (both equations are identical here). Thus y can be any real number, r say, and then $x = r$. Thus, the set of all eigenvectors with corresponding eigenvalue -3 is

$$\left\{ \begin{pmatrix} r \\ r \end{pmatrix} : r \in \mathbb{R} \;\middle|\; r \neq 0 \right\}.$$

Note that in this part and the previous one, we obtained a repeated eigenvalue. In the previous part, ignoring the zero vector, we obtained a 2-space of corresponding eigenvectors, while in this part (which shows more typical behaviour) we only obtain a 1-space of corresponding eigenvectors (ignoring the zero vector).

Practice Question 3. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 4 \end{pmatrix}$.

(a) The characteristic polynomial of A is

$$\begin{aligned} \det(A - x\mathrm{I}_3) &= \begin{vmatrix} 1-x & 2 & 0 \\ 2 & 1-x & 0 \\ 1 & 2 & 4-x \end{vmatrix} \\ &= (1-x)(1-x)(4-x) - 2(2(4-x)) + 0 \\ &= (4-x)((1-x)^2 - 4) = (4-x)(x^2 - 2x - 3) \\ &= -(x-4)(x-3)(x+1). \end{aligned}$$

Thus the eigenvalues of A are -1, 3 and 4.

(b) To determine an eigenvector with corresponding eigenvalue -1, we solve

$$(A + \mathrm{I}_3)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{that is} \quad \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 1 & 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which is equivalent to the following system of linear equations:

$$\left.\begin{aligned} 2x + 2y \qquad &= 0 \\ 2x + 2y \qquad &= 0 \\ x + 2y + 5z &= 0 \end{aligned}\right\}.$$

This system is equivalent to

$$\left.\begin{aligned} 2x + 2y \qquad &= 0 \\ 0 &= 0 \\ y + 5z &= 0 \end{aligned}\right\},$$

which (discarding the degenerate equation $0 = 0$) is equivalent to

$$\left.\begin{aligned} 2x + 2y \qquad &= 0 \\ y + 5z &= 0 \end{aligned}\right\},$$

which is a system of non-degenerate linear equations in echelon form.

Now z is the only non-leading variable, and so to obtain one solution with x, y, z not all equal to zero, we may take $z = 1$, and then we have $y = -5z = -5$, and $2x - 10 = 0$, so $x = 5$.

Thus $\begin{pmatrix} 5 \\ -5 \\ 1 \end{pmatrix}$ is an eigenvector, with corresponding eigenvalue -1.

[Any nonzero scalar multiple of the above vector is also correct.]

(c) From Part (b), we see that such an ℓ is the line through the origin and $(5, -5, 1)$, and so ℓ has vector equation $\mathbf{r} = \mu \begin{pmatrix} 5 \\ -5 \\ 1 \end{pmatrix}$.

[Of course it is acceptable for the parameter to be called λ instead of μ, although now we are mostly using λ to denote an eigenvalue.]

Practice Question 4. We have $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \lambda\mathbf{u} + \lambda\mathbf{v} = \lambda(\mathbf{u} + \mathbf{v})$, where the equality $A\mathbf{u} + A\mathbf{v} = \lambda\mathbf{u} + \lambda\mathbf{v}$ follows from the fact that $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of A, both with corresponding eigenvalue λ.
Thus, if $\mathbf{u} + \mathbf{v} \neq \mathbf{0}_n$, then $\mathbf{u} + \mathbf{v}$ is an eigenvector of A with corresponding eigenvalue λ. (And if $\mathbf{u} + \mathbf{v} = \mathbf{0}_n$ then $\mathbf{u} + \mathbf{v}$ is *not* an eigenvector of A, by definition.)

Practice Question 5. [My intention when setting this question was that S_θ corresponds to a reflexion in a line through O at (anticlockwise) angle $\theta/2$ to the (positive) x-axis, as per my preferred convention. But in lectures this year you had S_θ corresponding to a reflexion in a line through O at angle θ to the x-axis. This has no effect on the answer Part (c), and only a minimal effect on Part (b). It does, however, have some effect on Part (a).]

(a) We have

$$A = S_\theta S_0 = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} = R_{2\theta},$$

and so A represents the rotation through angle 2θ. Now

$$\begin{aligned} B = S_\pi S_\theta &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos(-2\theta) & -\sin(-2\theta) \\ \sin(-2\theta) & \cos(-2\theta) \end{pmatrix} = R_{-2\theta}, \end{aligned}$$

and so B represents the rotation through angle -2θ.

It is especially unfortunate that $S_\pi = S_0$ under the convention in force. It was my intention that S_π be the reflexion in the y-axis (the equivalent of $S_{\pi/2}$ from lectures), whereas S_0 is the reflexion in the x-axis. So let us answer the question as it was intended. We have

$$A = S_\theta S_0 = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = R_\theta,$$

and so A represents the rotation through angle θ. Now

$$\begin{aligned} B = S_\pi S_\theta &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = \begin{pmatrix} -\cos\theta & -\sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos(\pi-\theta) & -\sin(\pi-\theta) \\ \sin(\pi-\theta) & \cos(\pi-\theta) \end{pmatrix} = R_{\pi-\theta}, \end{aligned}$$

and so B represents the rotation through angle $\pi - \theta$.

(b) We can prove this by showing that $S_\theta S_\theta = \mathrm{I}_2$, as follows:

$$\begin{aligned} S_\theta S_\theta &= \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \\ &= \begin{pmatrix} (\cos 2\theta)^2 + (\sin 2\theta)^2 & 0 \\ 0 & (\sin 2\theta)^2 + (\cos 2\theta)^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{I}_2. \end{aligned}$$

Alternatively, we can apply the formula for the inverse of a 2×2 matrix. We have $\det S_\theta = -1$, and so

$$(S_\theta)^{-1} = -\begin{pmatrix} -\cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} = S_\theta.$$

[With my preferred convention for S_θ, the above proofs would be essentially the same. The only difference is that every occurence of 2θ in a matrix would get replaced by θ; thus $\cos 2\theta$ becomes $\cos\theta$, and so on.]

(c) [There are (at least) two ways of doing this. Both these ways use Part (b) that shows that a reflexion is its own inverse. The first proof also uses the fact (proved in the Week 8 lecture notes) that if A and B are invertible $n \times n$ matrices then $(AB)^{-1} = B^{-1}A^{-1}$. The second proof checks directly that the required properties to be an inverse hold.]

First proof: We have

$$(S_{\theta_1} S_{\theta_2} S_{\theta_3})^{-1} = (S_{\theta_3})^{-1}(S_{\theta_1} S_{\theta_2})^{-1} = (S_{\theta_3})^{-1}(S_{\theta_2})^{-1}(S_{\theta_1})^{-1} = S_{\theta_3} S_{\theta_2} S_{\theta_1}.$$

Second proof: We have

$$\begin{aligned}(S_{\theta_1}S_{\theta_2}S_{\theta_3})(S_{\theta_3}S_{\theta_2}S_{\theta_1}) &= S_{\theta_1}S_{\theta_2}S_{\theta_3}(S_{\theta_3})^{-1}(S_{\theta_2})^{-1}(S_{\theta_1})^{-1}\\ &= S_{\theta_1}S_{\theta_2}\mathrm{I}_2(S_{\theta_2})^{-1}(S_{\theta_1})^{-1} = S_{\theta_1}S_{\theta_2}(S_{\theta_2})^{-1}(S_{\theta_1})^{-1}\\ &= S_{\theta_1}\mathrm{I}_2(S_{\theta_1})^{-1} = S_{\theta_1}(S_{\theta_1})^{-1} = \mathrm{I}_2,\end{aligned}$$

and

$$\begin{aligned}(S_{\theta_3}S_{\theta_2}S_{\theta_1})(S_{\theta_1}S_{\theta_2}S_{\theta_3}) &= S_{\theta_3}S_{\theta_2}S_{\theta_1}(S_{\theta_1})^{-1}(S_{\theta_2})^{-1}(S_{\theta_3})^{-1}\\ &= S_{\theta_3}S_{\theta_2}\mathrm{I}_2(S_{\theta_2})^{-1}(S_{\theta_3})^{-1} = S_{\theta_3}S_{\theta_2}(S_{\theta_2})^{-1}(S_{\theta_3})^{-1}\\ &= S_{\theta_3}\mathrm{I}_2(S_{\theta_3})^{-1} = S_{\theta_3}(S_{\theta_3})^{-1} = \mathrm{I}_2.\end{aligned}$$

Practice Question 6. This topic was not lectured this year, but does (or will) appear in Chapter 10 of the online notes. From the lecture notes, rotations through θ about the x-axis and z-axis have matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

respectively. So $R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ and $R_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) We have $R_3 = R_2R_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

(b) $\det(R_3 - \mathrm{I}_3) = \begin{vmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = (-1)(-1) - 1(-1) = 0$ (or to show this determinant is 0, add the last two columns to the first column and so change the first column to $\mathbf{0}_3$). Hence 1 is an eigenvalue.

A corresponding eigenvector is obtained from any nonzero solution of the system of equations:

$$\left.\begin{aligned} -x + 0y + z &= 0 \\ x - y + 0z &= 0 \\ 0x + y - z &= 0 \end{aligned}\right\}.$$

So an eigenvector of R_3 corresponding to the eigenvalue 1 is $\mathbf{u}_1 := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (or any nonzero scalar multiple of this).

(c) Since $R_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, the direction of the axis of rotation of R_3 is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(d) We have

$$(R_3)^3 = \left(\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right) \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathrm{I}_3.$$

Hence applying the rotation R_3 three times gives the identity. So if R_3 is a rotation through angle θ we deduce that $3\theta = 2\pi k$ for some integer k. (Angles are only defined up to adding integer multiples of 2π.) Since R_3 itself is not the identity, we deduce that R_3 is a rotation through $\pm 2\pi/3$ (modulo integer multiples of 2π).

(Looking from $(1, 1, 1)$ towards $O = (0, 0, 0)$, this rotation is through an anticlockwise angle of $2\pi/3$.)

(e) Let $R_4 = R_1 R_2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$. Then $\det(R_4 - \mathrm{I}_3) = \begin{vmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{vmatrix} =$ $(-1)(+1) + 1(+1) = 0$, so $+1$ is an eigenvalue, and an eigenvector corresponding to this eigenvalue 1 is $\mathbf{u}_2 := \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ (or any nonzero scalar multiple of this). Thus $R_4 = R_1 R_2$ has an axis of rotation different from that for $R_2 R_1$. But a direct calculation of $(R_4)^3$ shows that $(R_4)^3 = \mathrm{I}_3 \neq R_4$, so R_4 is still a rotation through $\pm 2\pi/3$. (Looking from $(1, -1, 1)$ towards $O = (0, 0, 0)$, this rotation is through an anticlockwise angle of $2\pi/3$.)

Let $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ be square matrices. The *trace* of A, denoted $\operatorname{tr} A$, is the sum of the (top-left to bottom-right) diagonal entries of A. Thus $\operatorname{tr} A := \sum_{i=1}^{n} a_{ii}$. (We do not care about the top-right to bottom-left 'diagonal'.) The *order* of A, denoted $\mathrm{o}(A)$, is the least integer $m \geqslant 1$ such that $A^m = 1$. If no such m exists, then A has infinite order, and we write $\mathrm{o}(A) = \infty$. In the above both R_1 and R_2 have order 4, while both $R_1 R_2$ and $R_2 R_1$ have order 3.
We have seen that $AB \neq BA$ in general. Nonetheless, certain properties are shared by AB and BA, including determinant, trace, characteristic polynomial, order, and set of eigenvalues, though the corresponding eigenvectors are in general different.

For example, R_1R_2 and R_2R_1 both have determinant 1, trace 0, order 3, characteristic polynomial $1 - x^3$, and eigenvalue set $\{1, \frac{1}{2}(-1 + \mathrm{i}\sqrt{3}), \frac{1}{2}(-1 - \mathrm{i}\sqrt{3})\}$, where $\mathrm{i}^2 = -1$. However, the eigenvectors of R_1R_2 and R_2R_1 corresponding to the eigenvalue 1 differ, and cannot be made the same by scaling. In fact, $\mathbf{u}_1$ is not an eigenvector of R_1R_2 for any eigenvalue, and $\mathbf{u}_2$ is not an eigenvector of R_2R_1 for any eigenvalue

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