MTH4103 (2013–14)

Geometry I

Solutions 9

19th March 2014

Practice Question 1.

(a) The matrix representing s is $A = \begin{pmatrix} 1 & -1 & 5 \\ 3 & 4 & 0 \\ -2 & -1 & 1 \end{pmatrix}$. The matrix representing t is $B = \begin{pmatrix} 1 & -1 \\ 1 & 5 \\ -2 & 4 \end{pmatrix}$. The matrix representing $s \circ t$ is $AB = \begin{pmatrix} -10 & 14 \\ 7 & 17 \\ -5 & 1 \end{pmatrix}$.

(b) $s\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = A\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 5 \\ 3 & 4 & 0 \\ -2 & -1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$.

(c) $(s \circ t)\begin{pmatrix} -3 \\ 4 \end{pmatrix} = (AB)\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -10 & 14 \\ 7 & 17 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 86 \\ 47 \\ 19 \end{pmatrix}$.

(d) We have $s : \mathbb{R}^3 \to \mathbb{R}^3$ and $t : \mathbb{R}^2 \to \mathbb{R}^3$. Thus, for $\mathbf{u} \in \mathbb{R}^3$, we should have $(t \circ s)(\mathbf{u}) = t(s(\mathbf{u}))$, but $s(\mathbf{u}) \in \mathbb{R}^3$ and the domain of t is $\mathbb{R}^2$, and so $t \circ s$ does not make sense.

Practice Question 2. In this question, for consistency with lectures, S_θ is the matrix representing the reflexion in the line at (anticlockwise) angle θ to the (positive) x-axis, rather my preferred convention that the angle be $\theta/2$. The effect of this is that I have used $S_{3\pi/4}$, $S_{-\pi/4}$, S_0 and $S_{\pi/2}$ below instead of my preferred notation of $S_{3\pi/2}$, $S_{-\pi/2}$, S_0 and S_π respectively. The matrices obtained as the answers are identical under either convention.

(a) The line $y = -x$ is at angle $3\pi/4$ from the (positive) x-axis, and so a reflexion in this line is represented by

$$S_{3\pi/4} = \begin{pmatrix} \cos 2(\frac{3\pi}{4}) & \sin 2(\frac{3\pi}{4}) \\ \sin 2(\frac{3\pi}{4}) & -\cos 2(\frac{3\pi}{4}) \end{pmatrix} = \begin{pmatrix} \cos\frac{3\pi}{2} & \sin\frac{3\pi}{2} \\ \sin\frac{3\pi}{2} & -\cos\frac{3\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Note that you could have said that the angle is $-\pi/4$ instead, and so the reflexion is $S_{-\pi/4}$. Fortunately, we have $\cos(-\frac{\pi}{2}) = 0 = \cos\frac{3\pi}{2}$ and $\sin(-\frac{\pi}{2}) = -1 = \sin\frac{3\pi}{2}$, and so $S_{-\pi/4} = S_{3\pi/4}$.

Alternatively, you could equally well just observe that the reflexion sends $(1,0) \to (0,-1)$ and $(0,1) \to (-1,0)$, and deduce that the matrix has these as its columns.

(b) This rotation is represented by $R_{\pi/2} = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(c) [Note the order of multiplication!] This transformation is represented by

$$R_{\pi/2}S_{3\pi/4} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = S_0.$$

(d) [Note the order of multiplication!] This transformation is represented by

$$S_{3\pi/4}R_{\pi/2} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = S_{\pi/2}.$$

Feedback Question. If s and t are maps (functions), whether linear or not, from $\mathbb{R}^n$ to $\mathbb{R}^m$, we can define the maps $s+t$ and λs (where $\lambda \in \mathbb{R}$ is a scalar) by $(s+t)(\mathbf{x}) := s(\mathbf{x}) + t(\mathbf{x})$ and $(\lambda s)(\mathbf{x}) := \lambda(s(\mathbf{x}))$ for all $\mathbf{x} \in \mathbb{R}^n$. The maps $-s$ and $s-t$ are defined in a similar manner. If s and t are linear then so are $s+t$ and λs [proof exercise]. But if s and t are not linear then $s+t$ is unlikely to be (but can be occasionally). Thus to determine whether a map is linear (or not), we can split a map into its components and determine this for each one; this is useful for Parts (f) and (g).
A theorem of lectures tells us that a linear map $s : \mathbb{R}^n \to \mathbb{R}^m$ must satisfy $s(\mathbf{0}_n) = \mathbf{0}_m$ and $s(-\mathbf{v}) = -s(\mathbf{v})$ for all $\mathbf{v} \in \mathbb{R}^n$. So a (possible) way of proving that $s : \mathbb{R}^n \to \mathbb{R}^m$ is *not* linear is to show that $s(\mathbf{0}_n) \neq \mathbf{0}_m$, or to exhibit a vector $\mathbf{v}$ such that $s(-\mathbf{v}) \neq -s(\mathbf{v})$. Note however that the non-linear maps (s) of Parts (c), (d) and (e) satisfy $s(\mathbf{0}_n) = \mathbf{0}_m$, and it is even possible for a non-linear map $s : \mathbb{R}^n \to \mathbb{R}^m$ to satisfy $s(\mathbf{0}_n) = \mathbf{0}_m$ and $s(-\mathbf{v}) = -s(\mathbf{v})$ for all $\mathbf{v} \in \mathbb{R}^n$, such as the one in Part (c).

(a) A function $t : \mathbb{R}^n \to \mathbb{R}^m$ is a *linear transformation* if for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$ we have $t(\mathbf{u}+\mathbf{v}) = t(\mathbf{u}) + t(\mathbf{v})$ and $t(\alpha\mathbf{u}) = \alpha t(\mathbf{u})$.

(b) The function $t : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $t\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -4b \\ 3a+b \end{pmatrix}$ is **linear**. We prove this directly by checking the two rules that must be satisfied for t to be a linear transformation.

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, with $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, and let $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} t(\mathbf{u}+\mathbf{v}) &= t\begin{pmatrix} u_1+v_1 \\ u_2+v_2 \end{pmatrix} = \begin{pmatrix} -4(u_2+v_2) \\ 3(u_1+v_1)+(u_2+v_2) \end{pmatrix} \\ &= \begin{pmatrix} -4u_2-4v_2 \\ 3u_1+u_2+3v_1+v_2 \end{pmatrix} = \begin{pmatrix} -4u_2 \\ 3u_1+u_2 \end{pmatrix} + \begin{pmatrix} -4v_2 \\ 3v_1+v_2 \end{pmatrix} \\ &= t(\mathbf{u}) + t(\mathbf{v}) \end{aligned}$$

and

$$t(\alpha\mathbf{u}) = t\begin{pmatrix} \alpha u_1 \\ \alpha u_2 \end{pmatrix} = \begin{pmatrix} -4\alpha u_2 \\ 3\alpha u_1 + \alpha u_2 \end{pmatrix} = \begin{pmatrix} \alpha(-4u_2) \\ \alpha(3u_1+u_2) \end{pmatrix} = \alpha t(\mathbf{u}).$$

(c) The function $t : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $t\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a \\ b^3 \end{pmatrix}$ is **not linear**. One way to see this is set $\mathbf{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and note that $t(2\mathbf{u}) = 8\mathbf{u} = 8t(\mathbf{u}) \neq 2t(\mathbf{u})$.

(d) In this part, and the next, $\mathbb{R}$ is of course $\mathbb{R}^1$. The function $t : \mathbb{R}^3 \to \mathbb{R}$ defined by $t(\mathbf{r}) = |\mathbf{r}|$ is **not linear**, and nor is any function $t : \mathbb{R}^3 \to \mathbb{R}$ defined by $t(\mathbf{r}) = |\mathbf{r}|^m$, where $m \in \mathbb{N}$. We have $|\mathbf{i}| = |-\mathbf{i}| = 1$, and so, for any m, we have $t(\mathbf{i}) = t(-\mathbf{i}) = 1^m = 1$. Therefore, $t(-\mathbf{i}) \neq -t(\mathbf{i})$.

(e) The function $t : \mathbb{R}^3 \to \mathbb{R}$ defined by $t(\mathbf{r}) = \mathbf{r}\cdot\mathbf{r} = |\mathbf{r}|^2$ is **not linear**, since it is the case $m = 2$ of the previous part.

(f) The function $t : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $t(\mathbf{r}) = 2(\mathbf{i}\cdot\mathbf{r})\mathbf{k} - 3(\mathbf{r} \times \mathbf{j})$ is **linear**, and each "component" $\mathbf{r} \mapsto 2(\mathbf{i}\cdot\mathbf{r})\mathbf{k}$ and $\mathbf{r} \mapsto -3(\mathbf{r} \times \mathbf{j})$ is also linear. In fact, theorems much earlier in the course are tantamount to stating that the dot and cross products are linear in both variables, a property that is called *bilinearity*. (Note that both the dot and cross products are functions of two variables.)

So let us prove that t is linear. For all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ and all scalars $\alpha \in \mathbb{R}$ we have

$$\begin{aligned} t(\mathbf{u}+\mathbf{v}) &= 2(\mathbf{i}\cdot(\mathbf{u}+\mathbf{v}))\mathbf{k} - 3((\mathbf{u}+\mathbf{v}) \times \mathbf{j}) \\ &= 2((\mathbf{i}\cdot\mathbf{u}) + (\mathbf{i}\cdot\mathbf{v}))\mathbf{k} - 3((\mathbf{u} \times \mathbf{j}) + (\mathbf{v} \times \mathbf{j})) \\ &= (2(\mathbf{i}\cdot\mathbf{u})\mathbf{k} - 3(\mathbf{u} \times \mathbf{j})) + (2(\mathbf{i}\cdot\mathbf{v})\mathbf{k} - 3(\mathbf{v} \times \mathbf{j})) = t(\mathbf{u}) + t(\mathbf{v}) \end{aligned}$$

and

$$\begin{aligned} t(\alpha\mathbf{u}) &= 2(\mathbf{i}\cdot(\alpha\mathbf{u}))\mathbf{k} - 3((\alpha\mathbf{u})\times\mathbf{j}) = 2(\alpha(\mathbf{i}\cdot\mathbf{u}))\mathbf{k} - 3(\alpha(\mathbf{u}\times\mathbf{j})) \\ &= 2\alpha(\mathbf{i}\cdot\mathbf{u})\mathbf{k} - 3\alpha(\mathbf{u}\times\mathbf{j}) = \alpha(2(\mathbf{i}\cdot\mathbf{u})\mathbf{k} - 3(\mathbf{u}\times\mathbf{j})) = \alpha t(\mathbf{u}), \end{aligned}$$

and so t is linear. (Standard properties of $\cdot$ and $\times$ are used throughout.)

(g) The function $t : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $t(\mathbf{r}) = \mathbf{r}\times(2\mathbf{r}) + (\mathbf{i}\cdot\mathbf{r})\mathbf{r} + \mathbf{j} + (\mathbf{k}\times\mathbf{r})\times\mathbf{r}$ is **not linear**, since $t(\mathbf{0}_3) = \mathbf{j} \neq \mathbf{0}_3$. The only "component" of this that is linear (and even this superficially looks non-linear) is the map $\mathbf{r} \mapsto \mathbf{r}\times(2\mathbf{r})$, since $\mathbf{r}\times(2\mathbf{r}) = \mathbf{0}_3$ for all $\mathbf{r} \in \mathbb{R}^3$.

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