MTH4103 (2013–14)

Geometry I

Solutions 8

12th March 2014

Practice Question 1.

(a) Let $A = \begin{pmatrix} 7 & -4 \\ 5 & -3 \end{pmatrix}$. Then $\det A = -21 - (-20) = -1 \neq 0$, and so A is invertible. Thus $A^{-1} = \frac{1}{\det A}\begin{pmatrix} -3 & 4 \\ -5 & 7 \end{pmatrix} = -\begin{pmatrix} -3 & 4 \\ -5 & 7 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 5 & -7 \end{pmatrix}$.

(b) Let $A = \begin{pmatrix} 6 & -4 \\ -15 & 10 \end{pmatrix}$. Then $\det A = 60-60 = 0$, and so A is not invertible.

(c) Let $A = \begin{pmatrix} 4 & -3 \\ 3 & 2 \end{pmatrix}$. Then $\det A = 8 - (-9) = 17 \neq 0$, and so A is invertible. Thus $A^{-1} = \frac{1}{\det A}\begin{pmatrix} 2 & 3 \\ -3 & 4 \end{pmatrix} = \frac{1}{17}\begin{pmatrix} 2 & 3 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} \frac{2}{17} & \frac{3}{17} \\ -\frac{3}{17} & \frac{4}{17} \end{pmatrix}$.

Practice Question 2. (a) Let $A = \begin{pmatrix} 1 & -3 & 2 \\ -1 & 1 & -1 \\ -4 & -2 & -1 \end{pmatrix}$. Then

$$\det A = \begin{vmatrix} 1 & -1 \\ -2 & -1 \end{vmatrix} - (-1)\begin{vmatrix} -3 & 2 \\ -2 & -1 \end{vmatrix} + (-4)\begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix} = -3 + 7 - 4 = 0.$$

(b) Let $A = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 3 & 1 \\ 0 & 3 & -1 \end{pmatrix}$. Then

$$\det A = 4\begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} + 0\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = -24 - (-10) + 0 = -14.$$

Practice Question 3. By applying column operations to A (subtracting Column 3 from the other two) we see that:

$$\det A = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1(1) - 0 + 0 = 1.$$

Letting A_{ij} denote the 2×2 matrix obtained from A by deleting the i-th row and j-th column, and applying the formula for A^{-1} from the notes, we deduce that

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} \det A_{11} & -\det A_{21} & \det A_{31} \\ -\det A_{12} & \det A_{22} & -\det A_{32} \\ \det A_{13} & -\det A_{23} & \det A_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 3 \end{pmatrix}.$$

We may write the eqations together as the matrix equation $A\mathbf{x} = \mathbf{y}$, where $\mathbf{x}$ is the column vector (3×1 matrix) with entries x_1, x_2, x_3, and $\mathbf{y}$ is the column vector with entries y_1, y_2, y_3. Now $A\mathbf{x} = \mathbf{y}$ if and only if $A^{-1}A\mathbf{x} = A^{-1}\mathbf{y}$ if and only if $\mathrm{I}_n\mathbf{x} = A^{-1}\mathbf{y}$ if and only if $\mathbf{x} = A^{-1}\mathbf{y}$. So the (unique) solution for the equations for given y_1, y_2, y_3 is

$$x_1 = y_1 - y_3, \quad x_2 = y_2 - y_3, \quad x_3 = -y_1 - y_2 + 3y_3.$$

Practice Question 4. From lectures, we know that $(AB)^{-1} = B^{-1}A^{-1}$ and $(AB)^\mathsf{T} = A^\mathsf{T}B^\mathsf{T}$ for all invertible $n \times n$ matrices. So we have:

$$(AB)^{-\mathsf{T}} = ((AB)^{-1})^\mathsf{T} = (B^{-1}A^{-1})^\mathsf{T} = (A^{-1})^\mathsf{T}(B^{-1})^\mathsf{T} = A^{-\mathsf{T}}B^{-\mathsf{T}}.$$

Practice Question 5. First we note that A^T, B^T, $(A+B)^\mathsf{T}$, $(-A)^\mathsf{T}$, $-(A^\mathsf{T})$, $(\lambda A)^\mathsf{T}$ and $\lambda(A^\mathsf{T})$ are all $n \times m$ matrices, while both A and $(A^\mathsf{T})^\mathsf{T}$ are $m \times n$ matrices. So in all parts of the question, the two matrices we are trying to prove equal have the same size ($m \times n$ in Part (a), and $n \times m$ for the other parts).

(a) For all i and j (with $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$) we have

$$(i,j)\text{-entry of } (A^\mathsf{T})^\mathsf{T} = (j,i)\text{-entry of } (A^\mathsf{T}) = (i,j)\text{-entry of } A,$$

and so $(A^\mathsf{T})^\mathsf{T} = A$, since $(A^\mathsf{T})^\mathsf{T}$ and A have the same size ($m \times n$), and the same (i,j)-entry for all applicable i and j.

(b) For all i and j (with $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$) the (i,j)-entries of A^T, B^T and $(A+B)^\mathsf{T}$ are the same as the (j,i)-entries A, B and $A+B$ respectively, namely a_{ji}, b_{ji} and $a_{ji} + b_{ji}$. Therefore $A^\mathsf{T} + B^\mathsf{T}$ has (i,j)-entry $a_{ji} + b_{ji}$, which is also the (i,j)-entry of $(A+B)^\mathsf{T}$, and so $(A+B)^\mathsf{T} = A^\mathsf{T} + B^\mathsf{T}$.

(c) This is a specialisation of Part (d) to $\lambda = -1$, so please see the next part.

(d) For all i and j (with $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$) we have:

$$\begin{aligned}(i,j)\text{-entry of } (\lambda A)^{\mathsf{T}} &= (j,i)\text{-entry of } \lambda A = \lambda \times ((j,i)\text{-entry of } A)\\ &= \lambda \times ((i,j)\text{-entry of } A^{\mathsf{T}}) = (i,j)\text{-entry of } \lambda(A^{\mathsf{T}}),\end{aligned}$$

and so $(\lambda A)^{\mathsf{T}} = \lambda(A^{\mathsf{T}})$ for all $m \times n$ matrices A and all scalars λ.

Practice Question 6. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \text{ so that } A+B = \begin{pmatrix} a+p & b+q \\ c+r & d+s \end{pmatrix}.$$

The adjugates of A, B and $A+B$ are respectively as follows:

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \begin{pmatrix} s & -q \\ -r & p \end{pmatrix} \text{ and } \begin{pmatrix} d+s & -(b+q) \\ -(c+r) & a+p \end{pmatrix}.$$

It is now immediately apparent that $\operatorname{adj} A + \operatorname{adj} B = \operatorname{adj}(A+B)$.

If A and B are any 1×1 matrices then $\operatorname{adj} A = \operatorname{adj} B = \operatorname{adj}(A+B) = \mathrm{I}_1$, so that $\operatorname{adj}(A+B) = \mathrm{I}_1 \neq 2\mathrm{I}_1 = \operatorname{adj} A + \operatorname{adj} B$.

For the 3×3 case, we let $A = B = \mathrm{I}_3$. Then $\operatorname{adj} A = \operatorname{adj} B = \operatorname{adj}(A+B) = \mathrm{I}_3$, so that $\operatorname{adj}(A+B) = \operatorname{adj}(2\mathrm{I}_3) = 4\mathrm{I}_3 \neq 2\mathrm{I}_3 = \operatorname{adj} A + \operatorname{adj} B$. More generally, if we let $A = \alpha\mathrm{I}_3$ and $B = \beta\mathrm{I}_3$, where α and β are scalars, then $\operatorname{adj} A = \alpha^2\mathrm{I}_3$, and so we find that $\operatorname{adj}(A+B) \neq \operatorname{adj} A + \operatorname{adj} B$ if and only if $2\alpha\beta \neq 0$. Let E_{ij} denote the 3×3 matrix with 1 in the (i,j)-position, and 0's elsewhere. Then

$$\operatorname{adj} \mathrm{E}_{11} = \operatorname{adj} \mathrm{E}_{22} = \operatorname{adj} \mathrm{E}_{11} + \operatorname{adj} \mathrm{E}_{22} = 0_{3\times 3} \neq \mathrm{E}_{33} = \operatorname{adj}(\mathrm{E}_{11} + \mathrm{E}_{22}).$$

Some cases of equality in the 3×3 case are as follows. Let A and B be scalar multiples of the all-1's matrix $\mathrm{J}_3 = 1_{3\times 3}$. Then $\operatorname{adj} A = \operatorname{adj} B = \operatorname{adj}(A+B) = 0_{3\times 3} = \operatorname{adj} A + \operatorname{adj} B$.

Feedback Question.

(a) You should have provided a proof valid for $n \times n$ matrices (by applying the definition of invertibility in your lecture notes), and not just handle the case of 2×2 matrices (using determinants or otherwise). The relevant place to look is Definition 7.4 in Section 7.8 of the online notes.

This is true. If A is an invertible $n \times n$ matrix, then

$$(-2A)(-\tfrac{1}{2}A^{-1}) = (-2)(-\tfrac{1}{2})(AA^{-1}) = \mathrm{I}_n,$$

and

$$(-\tfrac{1}{2}A^{-1})(-2A) = (-\tfrac{1}{2})(-2)(A^{-1}A) = \mathrm{I}_n,$$

and so $-2A$ is invertible [and $(-2A)^{-1} = -\frac{1}{2}A^{-1}$, where the bracketing in the expression $-\frac{1}{2}A^{-1}$ should be understood as $-\frac{1}{2}(A^{-1})$, which is not the same as $(-\frac{1}{2}A)^{-1}$].

(b) This is false. For example, let $A = \mathrm{I}_2$ and $B = -\mathrm{I}_2$. Then both A and B are invertible ($\det A = \det B = 1 \neq 0$), but $A + B = 0_{2\times 2}$ is certainly not invertible ($\det 0_{2\times 2} = 0$).

This statement is false for $n \times n$ matrices for all $n \geqslant 1$. We can take $A = \mathrm{I}_n$ and $B = -\mathrm{I}_n$ so that both A and B are self-inverse (we have $A^2 = B^2 = \mathrm{I}_n$), but $A + B = 0_{n\times n}$ is not invertible since $C0_{n\times n} = 0_{n\times n}C = 0_{n\times n} \neq \mathrm{I}_n$ for all $n \times n$ matrices C. (We did not use determinants here, but note that I_n, $-\mathrm{I}_n$ and $0_{n\times n}$ have determinants 1, $(-1)^n$ and 0 respectively.)

(c) This is true. If A is an invertible 2×2 matrix, we have

$$\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det \mathrm{I}_2 = 1,$$

and so $\det A \neq 0$ and $\det(A^{-1}) = 1/\det A$.

(d) This is true. The matrix αA can be formed by multiplying each of the three columns of A by α, and, by Theorem 8.5 in the lecture notes, each multiplication of a column by α multiplies the determinant by α. Hence, $\det(\alpha A) = \alpha^3 \det A$.

[Note: It is acceptable to write $\det \alpha A$ for $\det(\alpha A)$, but it is not acceptable to write $\det -A$ for $\det(-A)$, nor to write $\det A + B$ for $\det(A + B)$.]

(e) This is false. Actually, $\det(\alpha A) = \alpha^2 \det A$ for all 2×2 matrices A. But $\alpha^2 \det A = \alpha \det A$ precisely when $\det A = 0$ or $\alpha \in \{0, 1\}$. So, for example, the inequality $\det(\alpha A) \neq \alpha^2 \det A$ **does not hold** when $\alpha = 1$ and $A = \mathrm{I}_2$.

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