# MTH4103 (2013–14)

# Geometry I

## Solutions 7

**5th March 2014**

---

**Practice Question 1.**

(a) $AB = \begin{pmatrix} 1 & 12 \\ -3 & -6 \end{pmatrix}$.

(b) $BA = \begin{pmatrix} 1 & 9 \\ -4 & -6 \end{pmatrix}$.

(c) $5A - 3BA = \begin{pmatrix} -5 & -15 \\ 5 & 5 \end{pmatrix} + \begin{pmatrix} -3 & -27 \\ 12 & 18 \end{pmatrix} = \begin{pmatrix} -8 & -42 \\ 17 & 23 \end{pmatrix}$

(d) 
$$(B + 3\mathrm{I}_2)^2 = \left(\begin{pmatrix} -4 & -3 \\ 1 & -3 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right)^2 = \begin{pmatrix} -1 & -3 \\ 1 & 0 \end{pmatrix}^2$$
$$= \begin{pmatrix} -1 & -3 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & -3 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ -1 & -3 \end{pmatrix}.$$

(e) Since ${\mathrm{I}_2}^{17} = \mathrm{I}_2$, we have
$$3A + {\mathrm{I}_2}^{17} - 5B = \begin{pmatrix} -3 & -9 \\ 3 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 20 & 15 \\ -5 & 15 \end{pmatrix} = \begin{pmatrix} 18 & 6 \\ -2 & 19 \end{pmatrix}.$$

**Practice Question 2.** [An educated trial and error approach should have produced matrices with the required properties, but not necessarily the ones below.]

(a) For example, take $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then $A \neq 0_{2\times 2}$, but $A^2 = 0_{2\times 2}$.

(b) For example, take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $AB = 0_{2\times 2}$, but $BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = A \neq 0_{2\times 2}$.

For Part (a) it is fun to find the general $2 \times 2$ matrix $A$ satisfying $A \neq 0_{2\times 2} = A^2$. So let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then we want

$$A^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ca + dc & d^2 + cb \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

but $(a, b, c, d) \neq (0, 0, 0, 0)$. Note that once we have determined the conditions satisfied by $a, b, c, d$, we must prove that there is a quadruple $(a, b, c, d)$ that satisfies these conditions, hence the explicit example above. The condition that $A^2 = 0_{2\times 2}$ means that we require

$$a^2 + bc = d^2 + bc = b(a + d) = c(a + d) = 0. \qquad (1)$$

If $a + d \neq 0$ then $b = c = 0$, and Equation 1 becomes $a^2 = d^2 = 0$, whence $a = d = 0$, which contradicts $a + d \neq 0$. Therefore $a + d = 0$, and thus $d = -a$, and now Equation 1 reduces to $a^2 + bc = 0$. So if $a = 0$ we get $b = 0$ or $c = 0$ (or both), and if $a \neq 0$ then $bc = -a^2 \neq 0$ so that $b, c \neq 0$ and so $c = -a^2/b$.
Therefore the possibilities for $(a, b, c, d)$ are $(0, 0, 0, 0)$, $(0, 0, \lambda, 0)$, $(0, \lambda, 0, 0)$ and $(\lambda, \mu, -\lambda^2/\mu, -\lambda)$, where $\lambda$ and $\mu$ are any nonzero numbers. In each of these cases we easily verify that $A^2 = 0_{2\times 2}$, while only in the first of these do we have $A = 0_{2\times 2}$. (We also easily verify that each of the other three cases do exist.)

**Practice Question 3.** [These are similar to proofs in your Week 8 lecture notes.]

(a) Now $-A$ has size $m \times n$, and $(i, j)$-entry $-a_{ij}$, so $A + (-A)$ has size $m \times n$, and $(i, j)$-entry $a_{ij} + (-a_{ij}) = 0$. Hence $A + (-A) = 0_{m\times n}$.

(b) Consider first the left-hand side. The matrix $(\alpha + \beta)A$ has size $m \times n$, and has $(i, j)$-entry

$$(\alpha + \beta)a_{ij} = \alpha a_{ij} + \beta a_{ij}.$$

Now consider the right-hand side. The matrix $\alpha A$ has size $m \times n$ and $(i, j)$-entry $\alpha a_{ij}$, the matrix $\beta A$ has size $m \times n$ and $(i, j)$-entry $\beta a_{ij}$, and so $\alpha A + \beta A$ has size $m \times n$ and $(i, j)$-entry

$$\alpha a_{ij} + \beta a_{ij}.$$

Hence $(\alpha + \beta)A = \alpha A + \beta A$.

(c) Consider first the left-hand side. The matrix $AX$ has size $m \times p$, and has $(i,j)$-entry $a_{i1}x_{1j} + a_{i2}x_{2j} + \cdots + a_{in}x_{nj}$. Therefore $\alpha(AX)$ has size $m \times p$ and $(i,j)$-entry

$$\alpha(a_{i1}x_{1j} + a_{i2}x_{2j} + \cdots + a_{in}x_{nj}).$$

Now consider the right-hand side. The matrix $\alpha X$ has size $n \times p$, and so the matrix $A(\alpha X)$ has size $m \times p$. Moreover, the $(i,j)$-entry of $A(\alpha X)$ is

$$\begin{aligned} a_{i1}(\alpha x_{1j}) + a_{i2}(\alpha x_{2j}) + \cdots + a_{in}(\alpha x_{nj}) &= \alpha a_{i1}x_{1j} + \alpha a_{i2}x_{2j} + \cdots + \alpha a_{in}x_{nj} \\ &= \alpha(a_{i1}x_{1j} + a_{i2}x_{2j} + \cdots + a_{in}x_{nj}). \end{aligned}$$

Hence $\alpha(AX) = A(\alpha X)$.

**Feedback Question.**

(a) The expression $A - 3B = A + (-3B)$ has no meaning since the size of $A$ $(3 \times 3)$ is not the same as the size of $-3B$ $(3 \times 2)$.

(b) $-A + 5C = \begin{pmatrix} 7 & 1 & -3 \\ -1 & 0 & 3 \\ -1 & -2 & 0 \end{pmatrix} + \begin{pmatrix} -10 & -10 & 15 \\ 10 & 5 & 0 \\ -5 & 5 & -10 \end{pmatrix} = \begin{pmatrix} -3 & -9 & 12 \\ 9 & 5 & 3 \\ -6 & 3 & -10 \end{pmatrix}.$

(c) $\begin{pmatrix} 1 & 5 & -2 \end{pmatrix} B = \begin{pmatrix} 1 & 5 & -2 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -3 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -16 & 5 \end{pmatrix}.$

(d) The expression $BA$ has no meaning since the number of columns of $B$ (2) is not equal to the number of rows of $A$ (3).

(e) $CB = \begin{pmatrix} 6 & -1 \\ 3 & -3 \\ -10 & 5 \end{pmatrix}.$

(f) $CA = \begin{pmatrix} 15 & 8 & 0 \\ -13 & -2 & 3 \\ 6 & -3 & -6 \end{pmatrix}.$

Dr John N. Bray, 5th March 2014