MTH4103 (2013–14)

Geometry I

Solutions 6

26th February 2014

Practice Question 1.
We have: $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = -((\mathbf{v}+\mathbf{w})\times\mathbf{u}) = -((\mathbf{v}\times\mathbf{u})+(\mathbf{w}\times\mathbf{u}))$, by assumed right-distributivity, and this is $-(-(\mathbf{u}\times\mathbf{v})-(\mathbf{u}\times\mathbf{w})) = (\mathbf{u}\times\mathbf{v})+(\mathbf{u}\times\mathbf{w})$.

Practice Question 2. [Draw a picture to help you visualise the situation, and hence to apply an appropriate theorem from your lecture notes.]
The vector $\mathbf{u}$ represented by $\overrightarrow{AB}$ is

$$\begin{pmatrix}1\\-1\\4\end{pmatrix} - \begin{pmatrix}1\\-3\\1\end{pmatrix} = \begin{pmatrix}0\\2\\3\end{pmatrix},$$

and the vector $\mathbf{v}$ represented by $\overrightarrow{AD}$ is also that represented by $\overrightarrow{BC}$, which is

$$\begin{pmatrix}4\\1\\0\end{pmatrix} - \begin{pmatrix}1\\-1\\4\end{pmatrix} = \begin{pmatrix}3\\2\\-4\end{pmatrix}.$$

By a theorem in the lecture notes, the area of parallelogram $ABCD$ is equal to

$$|\mathbf{u}\times\mathbf{v}| = |-14\mathbf{i}+9\mathbf{j}-6\mathbf{k}| = \sqrt{(-14)^2+9^2+(-6)^2} = \sqrt{313}.$$

Note that you obtain the same answer if you (incorrectly) use the cross product of the vectors represented by $\overrightarrow{AB}$ and $\overrightarrow{AC}$ instead $\overrightarrow{AB}$ and $\overrightarrow{AD}$. Indeed, the cross products themselves are actually equal. See the next question for why this 'incorrect' formula always (inadvertently) gives you the correct answer.

Practice Question 3. We suppose that $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ represent $\mathbf{u}$, $\mathbf{w}$ and $\mathbf{v}$ respectively, so that $\mathbf{u}=\mathbf{b}-\mathbf{a}$, $\mathbf{w}=\mathbf{c}-\mathbf{a}$ and $\mathbf{v}=\mathbf{d}-\mathbf{a}$, with the notation chosen so that $\mathbf{u}$ and $\mathbf{v}$ correspond to the sides of the parallelogram $ABCD$ and $\mathbf{w}$ corresponds to a diagonal. From lecture notes, the area of the parallelogram is $|\mathbf{u}\times\mathbf{v}|$. Now $\mathbf{w}=\mathbf{u}+\mathbf{v}$, and so $\mathbf{v}=\mathbf{w}-\mathbf{u}$, and so the area is

$$|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}\times(\mathbf{w}-\mathbf{u})| = |(\mathbf{u}\times\mathbf{w})-(\mathbf{u}\times\mathbf{u})| = |(\mathbf{u}\times\mathbf{w})-\mathbf{0}| = |\mathbf{u}\times\mathbf{w}|,$$

and so in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ the area is $|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})|$.

Practice Question 4. Using the Cartesian equations for the line, ℓ say, we set

$$\lambda = \frac{x-4}{3} = \frac{y+2}{3} = \frac{z-1}{-5},$$

giving $x = 4 + 3\lambda$, $y = -2 + 3\lambda$, $z = 1 - 5\lambda$, and so the line has vector equation $\mathbf{r} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix}$. Let $P = (4, -2, 1)$, $Q = (-2, 1, 5)$, $\mathbf{u} = \begin{pmatrix} 3 \\ 3 \\ -5 \end{pmatrix}$, and $\mathbf{v}$ be represented by $\overrightarrow{PQ}$. Then $\mathbf{v} = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \\ 4 \end{pmatrix}$. Note that P is on ℓ, and we want to find the distance from Q to ℓ. Thus we calculate that

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} 3 & 3 \\ -5 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & -6 \\ -5 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -6 \\ 3 & 3 \end{vmatrix} \mathbf{k} = 27\mathbf{i} + 18\mathbf{j} + 27\mathbf{k},$$

and so the distance from Q to the line ℓ is

$$\frac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{u}|} = \frac{\sqrt{729 + 324 + 729}}{\sqrt{9 + 9 + 25}} = \frac{\sqrt{1782}}{\sqrt{43}} = \frac{9\sqrt{22}}{\sqrt{43}}.$$

Practice Question 5.

(a) These lines are parallel (since their direction vectors are parallel), and so the distance between them is the distance between one, say the first, namely $\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$, and a point P, say $(5, 2, 3)$, on the other. Let $A = (4, -1, -3)$, $P = (5, 2, 3)$, $\mathbf{u} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$, and $\mathbf{v}$ be represented by $\overrightarrow{AP}$.

Then $\mathbf{v} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$, and therefore

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} -3 & 3 \\ 1 & 6 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 1 & 6 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ -3 & 3 \end{vmatrix} \mathbf{k} = -21\mathbf{i} - 11\mathbf{j} + 9\mathbf{k},$$

and the so distance from P to the (first) line is

$$\frac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{u}|} = \frac{\sqrt{441 + 121 + 81}}{\sqrt{4 + 9 + 1}} = \frac{\sqrt{643}}{\sqrt{14}}.$$

(b) Let $\mathbf{a} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}$. Then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} -3 & 7 \\ 1 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ -3 & 7 \end{vmatrix} \mathbf{k} = -16\mathbf{i} - 5\mathbf{j} + 17\mathbf{k},$$

and so the distance between the lines is

$$\frac{|(\mathbf{b} - \mathbf{a})\cdot(\mathbf{u} \times \mathbf{v})|}{|\mathbf{u} \times \mathbf{v}|} = \left| \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} -16 \\ -5 \\ 17 \end{pmatrix} \right| \Big/ \sqrt{256 + 25 + 289} = \frac{71}{\sqrt{570}}.$$

Practice Question 6. If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ then $|\mathbf{a} \times \mathbf{b}| = \sqrt{|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2} = 0$, since $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ and $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|^2|\mathbf{b}|^2 = 0$. Else, in the general case, we let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. We have $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta \geqslant 0$, and $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$. Thus, we have:

$$\begin{aligned} \sqrt{|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2} &= \sqrt{|\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta} \\ &= \sqrt{|\mathbf{a}|^2|\mathbf{b}|^2(1 - \cos^2\theta)} = \sqrt{|\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta} \\ &= |\mathbf{a}||\mathbf{b}||\sin\theta| = |\mathbf{a} \times \mathbf{b}|, \end{aligned}$$

where the last equality follows from the fact that $\sin\theta \geqslant 0$ for $0 \leqslant \theta \leqslant \pi$. It is also possible, but not very pretty, to use coördinates to do this question.

Practice Question 7. [For Part (b), you should note when I have used an explicit (counter-)example, and when I have proved something for all quaternions A, B, C, using symbolic computations.]

(a) Using various properties of the dot and cross products, we have:

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} &= -(\mathbf{c} \times (\mathbf{a} \times \mathbf{b})) = -((\mathbf{c}\cdot\mathbf{b})\mathbf{a} - (\mathbf{c}\cdot\mathbf{a})\mathbf{b}) \\ &= (\mathbf{c}\cdot\mathbf{a})\mathbf{b} - (\mathbf{c}\cdot\mathbf{b})\mathbf{a} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}. \end{aligned}$$

(b) To see that quaternion multiplication is non-commutative, we exhibit an explicit example when the product does not commute. Let A and B be the quaternions $(1, \mathbf{i})$ and $(0, \mathbf{j})$ respectively. We have $AB = (0, \mathbf{j} + \mathbf{k})$, whereas $BA = (0, \mathbf{j} - \mathbf{k})$. [Note that there are pairs of quaternions A and B such that $AB = BA$, for example $A = (1, -\mathbf{i})$ and $B = (2, 3\mathbf{i})$ when we have $AB = BA = (5, -\mathbf{i})$.]

For associativity, we let $A = (\alpha, \mathbf{a})$, $B = (\beta, \mathbf{b})$ and $C = (\gamma, \mathbf{c})$ be general quaternions. We evaluate $A(BC)$ and $(AB)C$ as follows:

$$\begin{aligned} A(BC) &= (\alpha, \mathbf{a}).((\beta, \mathbf{b}).(\gamma, \mathbf{c})) \\ &= (\alpha, \mathbf{a}).(\beta\gamma - \mathbf{b}\cdot\mathbf{c}, \beta\mathbf{c} + \gamma\mathbf{b} + \mathbf{b} \times \mathbf{c}) \\ &= (\alpha\beta\gamma - \alpha(\mathbf{b}\cdot\mathbf{c}) - \beta(\mathbf{a}\cdot\mathbf{c}) - \gamma(\mathbf{a}\cdot\mathbf{b}) - \mathbf{a}\cdot(\mathbf{b} \times \mathbf{c}), \\ &\qquad \alpha\beta\mathbf{c} + \alpha\gamma\mathbf{b} + \alpha(\mathbf{b} \times \mathbf{c}) + \beta\gamma\mathbf{a} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a} \\ &\qquad +\beta(\mathbf{a} \times \mathbf{c}) + \gamma(\mathbf{a} \times \mathbf{b}) + \mathbf{a} \times (\mathbf{b} \times \mathbf{c})) \end{aligned}$$

and

$$\begin{aligned} (AB)C &= ((\alpha, \mathbf{a}).(\beta, \mathbf{b})).(\gamma, \mathbf{c}) \\ &= (\alpha\beta - \mathbf{a}\cdot\mathbf{b}, \alpha\mathbf{b} + \beta\mathbf{a} + \mathbf{a} \times \mathbf{b}).(\gamma, \mathbf{c}) \\ &= (\alpha\beta\gamma - \gamma(\mathbf{a}\cdot\mathbf{b}) - \alpha(\mathbf{b}\cdot\mathbf{c}) - \beta(\mathbf{a}\cdot\mathbf{c}) - (\mathbf{a} \times \mathbf{b})\cdot\mathbf{c}, \\ &\qquad \alpha\beta\mathbf{c} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} + \gamma\alpha\mathbf{b} + \gamma\beta\mathbf{a} + \gamma(\mathbf{a} \times \mathbf{b}) \\ &\qquad +\alpha(\mathbf{b} \times \mathbf{c}) + \beta(\mathbf{a} \times \mathbf{c}) + (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}). \end{aligned}$$

The scalar components of $A(BC)$ and $(AB)C$ are equal since $\mathbf{a}\cdot(\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b})\cdot\mathbf{c}$ for all vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Comparing the vector components, we see that they are equal if and only if $-(\mathbf{b}\cdot\mathbf{c})\mathbf{a} + \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -(\mathbf{a}\cdot\mathbf{b})\mathbf{c} + (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$. But both sides of this equality are $-(\mathbf{b}\cdot\mathbf{c})\mathbf{a} + (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$, hence quaternion multiplication is associative.

Feedback Question.

(a) Let $P = (2, 5, 1)$, $Q = (-5, 2, -1)$, $R = (1, -2, -4)$, let $\mathbf{u}$ be represented by $\overrightarrow{PQ}$ and $\mathbf{v}$ be represented by $\overrightarrow{PR}$. Then $\mathbf{u} = \begin{pmatrix} -7 \\ -3 \\ -2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ -7 \\ -5 \end{pmatrix}$.

We require a Cartesian equation for the plane through $P = (2, 5, 1)$ and parallel to $\mathbf{u}$ and $\mathbf{v}$. A vector equation for this plane is $\mathbf{r}\cdot\mathbf{n} = \mathbf{p}\cdot\mathbf{n}$, where $\mathbf{n} = \mathbf{u} \times \mathbf{v}$ and $\mathbf{p}$ is the position vector of P. We can take $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \mathbf{i} - 33\mathbf{j} + 46\mathbf{k}$.

Now $\mathbf{p}\cdot\mathbf{n} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -33 \\ 46 \end{pmatrix} = -117$, so a vector equation for the plane is $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -33 \\ 46 \end{pmatrix} = -117$, and a Cartesian equation is $x - 33y + 46z = -117$.

Note: Any nonzero scalar multiples of the above equations are correct.

(b) Let $P = (3, -1, -2)$, $Q = (4, 1, -3)$, $R = (2, 3, -1)$, let $\mathbf{u}$ be represented by $\overrightarrow{PQ}$ and $\mathbf{v}$ be represented by $\overrightarrow{PR}$. Then $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$.

We require a Cartesian equation for the plane through $A = (-4, 2, 1)$ and parallel to $\mathbf{u}$ and $\mathbf{v}$. A vector equation for this plane is $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$, where $\mathbf{n} = \mathbf{u}\times\mathbf{v}$ and $\mathbf{a}$ is the position vector of A. We can take $\mathbf{n} = \mathbf{u}\times\mathbf{v} = 6\mathbf{i}+6\mathbf{k}$. In fact, we can take $\mathbf{n}$ to be *any* nonzero scalar multiple of $\mathbf{u} \times \mathbf{v}$, and so we choose $\mathbf{n} = \frac{1}{6}(\mathbf{u} \times \mathbf{v}) = \mathbf{i} + \mathbf{k}$ instead.

Now $\mathbf{a}\cdot\mathbf{n} = \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = -3$, and so a vector equation for the plane is $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = -3$, and a Cartesian equation is $x + z = -3$.

Note: Any nonzero scalar multiples of the above equations are correct. With $\mathbf{n} = \mathbf{u} \times \mathbf{v}$, we would have algorithmically obtained the Cartesian equation $6x + 6z = -18$, which is of course equivalent to $x + z = -3$. The points P, Q and R lie on the plane $x + z = 1$.

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