MTH4103 (2013–14)

Geometry I

Solutions 5

12th February 2014

Practice Question 1.

(a) The line ℓ through the points $(1,5,-1)$ and $(2,3,1)$ is the line through the point $(2,3,1)$ in the direction of the vector $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$. (It does not matter in which order we list the points on ℓ; I chose to swap them. Nor does it matter the I've chosen 'B' rather than 'A' as my favourite point on ℓ.) Thus, parametric equations for ℓ are

$$\left.\begin{aligned} x &= 2 + \lambda \\ y &= 3 - 2\lambda \\ z &= 1 + 2\lambda \end{aligned}\right\}.$$

We find the intersection of the plane Π with ℓ by solving

$$-6(2+\lambda) + (3-2\lambda) + 4(1+2\lambda) = 7$$

for λ. Simplifying, this gives $-5 = 7$, and thus $0 = 12$, a degenerate equation with no solutions. Thus ℓ intersects the plane Π in no points. Alternatively, as a set of points, the intersection is $\varnothing$, the empty set.

(b) Parametric equations for ℓ are

$$\left.\begin{aligned} x &= -3 + 2\lambda \\ y &= 1 - \lambda \\ z &= 1 + \lambda \end{aligned}\right\},$$

and parameteric equations for $\mathfrak{m}$ are

$$\left.\begin{aligned} x &= 5 + \mu \\ y &= -6 - 2\mu \\ z &= - 2\mu \end{aligned}\right\}.$$

To determine the intersection of ℓ with $\mathfrak{m}$ we solve

$$\left.\begin{aligned} -3 + 2\lambda &= 5 + \mu \\ 1 - \lambda &= -6 - 2\mu \\ 1 + \lambda &= 0 - 2\mu \end{aligned}\right\}$$

for λ and μ. We obtain the following system of linear equations:

$$\left.\begin{array}{rl} 2\lambda - \mu = & 8 \\ -\lambda + 2\mu = & -7 \\ \lambda + 2\mu = & -1 \end{array}\right\}.$$

We apply Gaußian elimination to this system. Elimination of λ from the second and third equations gives:

$$\left.\begin{array}{rl} 2\lambda - \mu = & 8 \\ \frac{3}{2}\mu = & -3 \\ \frac{5}{2}\mu = & -5 \end{array}\right\},$$

and the final echelon form, after discarding the degenerate equation $0 = 0$ is:

$$\left.\begin{array}{rl} 2\lambda - \mu = & 8 \\ \frac{3}{2}\mu = & -3 \end{array}\right\},$$

Applying back substitution, we find that $\mu = -2$, $\lambda = (8 + (-2))/2 = 3$ is the only solution.

Substituting $\lambda = 3$ into the parametric equations for ℓ, we get $x = 3$, $y = -2$, $z = 4$. Alternatively, and providing a check, substituting $\mu = -2$ into the parametric equations for $\mathfrak{m}$, we also get $x = 3$, $y = -2$, $z = 4$.

Thus ℓ and $\mathfrak{m}$ intersect in the single point $(3, -2, 4)$. Alternatively, as a set of points, the intersection is $\{(3, -2, 4)\}$.

Practice Question 2.

(a) $(\mathbf{k} \times \mathbf{i}) \times \mathbf{i} = \mathbf{j} \times \mathbf{i} = -\mathbf{k}$.

(b) $\mathbf{k} \times (\mathbf{i} \times \mathbf{i}) = \mathbf{k} \times \mathbf{0} = \mathbf{0}$.
[The answer is the zero vector $\mathbf{0}$, not the number 0.]

(c) $(3\mathbf{i} \times \mathbf{k}) \times (\mathbf{j} \times (-4\mathbf{i})) = 3(\mathbf{i} \times \mathbf{k}) \times (-4)(\mathbf{j} \times \mathbf{i}) = 3(-\mathbf{j}) \times (-4)(-\mathbf{k}) = (-3\mathbf{j}) \times (4\mathbf{k}) = (-12)(\mathbf{j} \times \mathbf{k}) = -12\mathbf{i}$.

(d) $(4\mathbf{k} \times \mathbf{i}) \cdot \mathbf{j} = (4(\mathbf{k} \times \mathbf{i})) \cdot \mathbf{j} = (4\mathbf{j}) \cdot \mathbf{j} = 4$.
[The answer is the number 4, not a vector.]

Practice Question 3. [The proof is obtained by careful appropriate modification of the proof in the lecture notes that $(\alpha\mathbf{u}) \times \mathbf{v} = \alpha(\mathbf{u} \times \mathbf{v})$.]
If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ or $\mathbf{u}$, $\mathbf{v}$ parallel or $\alpha = 0$, then $\mathbf{u} \times (\alpha\mathbf{v}) = \mathbf{0} = \alpha(\mathbf{u} \times \mathbf{v})$.

Otherwise, let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$, and let $\mathbf{w} = \mathbf{u} \times \mathbf{v}$. The angle between $\mathbf{u}$ and $\alpha\mathbf{v}$ is θ if $\alpha > 0$ and $\pi - \theta$ if $\alpha < 0$, and we have

$$|\alpha\mathbf{w}| = |\alpha||\mathbf{w}| = |\alpha||\mathbf{u}||\mathbf{v}| \sin\theta = |\mathbf{u}||\alpha\mathbf{v}| \sin\theta = |\mathbf{u}||\alpha\mathbf{v}| \sin(\pi - \theta).$$

Thus, we have $|\alpha\mathbf{w}| = |\mathbf{u} \times (\alpha\mathbf{v})|$, whether $\alpha > 0$ or $\alpha < 0$. Moreover, $\alpha\mathbf{w}$ is orthogonal to $\mathbf{u}$ and $\alpha\mathbf{v}$ (since $\mathbf{w}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$), and $\mathbf{u}$, $\alpha\mathbf{v}$, $\alpha\mathbf{w}$ is a right-handed triple (whether $\alpha > 0$ or $\alpha < 0$), since $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is a right-handed triple. Thus $\mathbf{u} \times (\alpha\mathbf{v}) = \alpha\mathbf{w} = \alpha(\mathbf{u} \times \mathbf{v})$.

Practice Question 4. (a) We have that

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \times \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3(3) - (-2)(-2) \\ -2(2) - 1(3) \\ 1(-2) - 3(2) \end{pmatrix} = \begin{pmatrix} 5 \\ -7 \\ -8 \end{pmatrix}.$$

An expansion using determinants is perfectly acceptable, and probably less prone to error. The triple scalar product of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is

$$\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w}) = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -7 \\ -8 \end{pmatrix} = 5 + 7 + (-24) = -12,$$

and the so required volume is $|\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w})| = |-12| = 12$.

(b) $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are not coplanar, since their triple scalar product is -12, which is nonzero. In fact, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is a left-handed triple, since their triple scalar product is (strictly) negative.

Practice Question 5. This question deals with a result for vector products that is analagous to one for scalar products that we handled in the Feedback Question of Sheet 2. However, unlike the dot product case, we cannot take $\mathbf{b} = \mathbf{a}$, since $\mathbf{a} \times \mathbf{a} = \mathbf{0}$. Instead, we go for a proof by contradiction. Suppose $\mathbf{a} \neq \mathbf{0}$ and that $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ for all vectors $\mathbf{b}$. Then we choose $\mathbf{b}$ to be a nonzero vector orthogonal to $\mathbf{a}$ (it is intuitive geometrically that such a vector exists). The angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{2}$, and $\sin\frac{\pi}{2} = 1$, so we get $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\frac{\pi}{2} = |\mathbf{a}||\mathbf{b}| \neq 0$. Thus $\mathbf{a} \times \mathbf{b} \neq \mathbf{0}$, a contradiction that establishes the result.

Alternatively, we can use coördinates. Let $\mathbf{a} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. We now calculate that $\mathbf{a} \times \mathbf{i} = c\mathbf{j} - b\mathbf{k}$, $\mathbf{a} \times \mathbf{j} = -c\mathbf{i} + a\mathbf{k}$ and $\mathbf{a} \times \mathbf{k} = b\mathbf{i} - a\mathbf{j}$. Any two of these cross products suffices to establish that $a = b = c = 0$, and thus that $\mathbf{a} = \mathbf{0}$. This contrasts with the dot product case, whereby all three dot products $\mathbf{a}\cdot\mathbf{i}$, $\mathbf{a}\cdot\mathbf{j}$ and $\mathbf{a}\cdot\mathbf{k}$ were required to show that $\mathbf{a} = \mathbf{0}$.

The case when $\mathbf{b} \times \mathbf{a} = \mathbf{0}$ for all vectors $\mathbf{b}$ can be proved similarly. Or we can simply note that if $\mathbf{b} \times \mathbf{a} = \mathbf{0}$ for all $\mathbf{b}$ then $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a}) = -\mathbf{0} = \mathbf{0}$ for all $\mathbf{b}$, and use the result we have already proved.

Feedback Question.

(a) This is false. For example, let $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{j}$, and $\mathbf{w} = \mathbf{j} + \mathbf{k}$. Then $\mathbf{u}$ is orthogonal to $\mathbf{v}$. Moreover,

$$\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w}) = \mathbf{i}\cdot(\mathbf{j} \times (\mathbf{j} + \mathbf{k})) = \mathbf{i}\cdot((\mathbf{j} \times \mathbf{j}) + (\mathbf{j} \times \mathbf{k})) = \mathbf{i}\cdot(\mathbf{0} + \mathbf{i}) = \mathbf{i}\cdot\mathbf{i} = 1 > 0,$$

and so $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is a right-handed triple. However, $\mathbf{v}$ is *not* orthogonal to $\mathbf{w}$, since

$$\mathbf{v}\cdot\mathbf{w} = \mathbf{j}\cdot(\mathbf{j} + \mathbf{k}) = \mathbf{j}\cdot\mathbf{j} + \mathbf{j}\cdot\mathbf{k} = 1 + 0 = 1 \neq 0.$$

How might one obtain this counterexample? When trying to construct a counterexample, it is valid to *guess* values for certain of the parameters, and then to see how far we can get with constructing a counterexample. (Lack of success *does not* guarantee the non-existence of a counterexample.) The only obvious restriction on $\mathbf{u}$ is that $\mathbf{u} \neq \mathbf{0}$ (since $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is not coplanar), so we try $\mathbf{u} = \mathbf{i}$. Then $\mathbf{v}$ must be nonzero and orthogonal to $\mathbf{u} = \mathbf{i}$, so we might as well try $\mathbf{v} = \mathbf{j}$. Instead of trying to guess a value for $\mathbf{w}$, we let $\mathbf{w} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, and determine how our conditions restrict a, b and c. The condition that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be right-handed is precisely the condition that $c > 0$. Since we are seeking a counterexample, we want $\mathbf{v}$ and $\mathbf{w}$ *not* to be orthogonal, so we need $\mathbf{v}\cdot\mathbf{w} \neq 0$, that is $b \neq 0$. Therefore, any vector $\mathbf{w} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ with $b \neq 0$ and $c > 0$ completes the counterexample we seek.

(b) This is false. For example, we have $(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = \mathbf{k} \times \mathbf{j} = -\mathbf{i}$, whereas $\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = \mathbf{i} \times \mathbf{0} = \mathbf{0}$. (This counterexample is easily obtained by letting $\mathbf{u}$ and $\mathbf{v}$ be distinct elements of $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.)

(c) This is true. Using two results from lectures, we get that $\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = (-1)(\mathbf{u} \times \mathbf{v})$ for all vectors $\mathbf{u}$ and $\mathbf{v}$. Therefore (using yet another result from lectures):

$$\begin{aligned}(\mathbf{u} \times \mathbf{v}) \times \mathbf{u} &= ((-1)(\mathbf{v} \times \mathbf{u})) \times \mathbf{u} = (-1)((\mathbf{v} \times \mathbf{u}) \times \mathbf{u}) \\ &= (-1)((-1)(\mathbf{u} \times (\mathbf{v} \times \mathbf{u}))) = ((-1)(-1))(\mathbf{u} \times (\mathbf{v} \times \mathbf{u})) \\ &= 1(\mathbf{u} \times (\mathbf{v} \times \mathbf{u})) = \mathbf{u} \times (\mathbf{v} \times \mathbf{u}),\end{aligned}$$

for all vectors $\mathbf{u}$ and $\mathbf{v}$, as required.

(d) This is true. Suppose $\mathbf{u}\times\mathbf{v} = \mathbf{v}\times\mathbf{u}$. Since $\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v})$, we have that $\mathbf{u}\times\mathbf{v} = -(\mathbf{u}\times\mathbf{v})$. Hence (adding $\mathbf{u}\times\mathbf{v}$ to both sides), we have $2(\mathbf{u}\times\mathbf{v}) = \mathbf{0}$, and so (scalar multiplying both sides by $\frac{1}{2}$), we have $\mathbf{u} \times \mathbf{v} = \mathbf{0}$. (The last sentence contains contains a proof that $-\mathbf{w} = \mathbf{w}$ implies that $\mathbf{w} = \mathbf{0}$; the vector $\mathbf{w}$ corresponds to the quantity $\mathbf{u} \times \mathbf{v}$ of the previous sentence.)

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