MTH4103 (2013–14)

Geometry I

Queen Mary
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Solutions 4

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As usual, you should show clearly all the steps in your calculations. You should perform Gaußian elimination and back substitution in **exactly** the same order as in these solutions. There are no choices in the procedures I have given you.

Practice Question 1. Proof of (b): Suppose $x = p$, $y = q$, $z = r$ is a solution of $(**)$. This means that $ap+bq+cr = d$ and $(e+\alpha a)p+(f+\alpha b)q+(g+\alpha c)r = h+\alpha d$. In particular, $x = p$, $y = q$, $z = r$ is a solution of the first equation of $(*)$. What about the second equation of $(*)$? We have

$$\begin{aligned} ep + fq + gr &= (e + \alpha a)p + (f + \alpha b)q + (g + \alpha c)r - \alpha(ap + bq + cr) \\ &= (h + \alpha d) - \alpha d = h, \end{aligned}$$

and so $x = p$, $y = q$, $z = r$ is a solution to the second equation of $(*)$ as well as the first, and we are done. The proof of (a) is very similar to the proof of (b).

Practice Question 2. (a) All variables of this system of non-degenerate linear equations in echelon form are leading variables. We have: $z = \frac{1}{4}$, $y-\frac{1}{4} = 1$, giving $y = 1\frac{1}{4} = \frac{5}{4}$, and finally $3x+2(\frac{5}{4})-5(\frac{1}{4}) = -3$, which gives $3x = -\frac{17}{4}$, and so $x = -\frac{17}{12}$.

Thus, the only solution to the system is $x = -\frac{17}{12}$, $y = \frac{5}{4}$, $z = \frac{1}{4}$.

(b) The non-leading variables of this system in echelon form of non-degenerate linear equations in x, y, z are x and z, and so these can take arbitrary real values, say $x = s$ and $z = t$. Then $7y + 3t = -4$, and so $7y = -4 - 3t$, whence $y = -\frac{4}{7} - \frac{3}{7}t$.

Thus, the solutions are $x = s$, $y = -\frac{4}{7} - \frac{3}{7}t$, $z = t$, where s and t can be any real numbers.

Practice Question 3. (a) We obtain the intersection by solving the following system of linear equations:

$$\left.\begin{aligned} 3x - 4y + 7z &= 2 \\ -6x + 8y - 14z &= 4 \end{aligned}\right\}.$$

Adding 2 times the first equation to the second, we get:

$$\left.\begin{array}{r} 3x - 4y + 7z = 2 \\ 0 = 8 \end{array}\right\} .$$

There is no solution to the system. Thus, the intersection of the planes is $\varnothing$, the empty set.

(b) We obtain the intersection by solving the following system of linear equations:

$$\left.\begin{array}{rcr} 3x - 4y + 7z &=& 2 \\ -6x + 8y - 14z &=& -4 \end{array}\right\} .$$

Adding 2 times the first equation to the second, we get:

$$\left.\begin{array}{r} 3x - 4y + 7z = 2 \\ 0 = 0 \end{array}\right\} .$$

Throwing away the equation $0 = 0$, we get:

$$3x - 4y + 7z = 2 \; \} .$$

which is a system in echelon form with no degenerate equations.

The non-leading variables of this system are y and z, and so y and z can take any real values, say $y = s$ and $z = t$. Then $x = (2+4s-7t)/3 = \frac{2}{3}+\frac{4}{3}s-\frac{7}{3}t$. The intersection is thus

$$\{\, (\tfrac{2}{3} + \tfrac{4}{3}s - \tfrac{7}{3}t, s, t) : s, t \in \mathbb{R} \,\}.$$

This intersection is the whole plane Π, which has equation $3x-4y+7z = 2$.

(c) We obtain the intersection by solving the following system of linear equations:

$$\left.\begin{array}{r} 3x - 4y + 7z = 2 \\ -6x + 8y - 11z = 8 \end{array}\right\} .$$

Adding 2 times the first equation to the second, we get:

$$\left.\begin{array}{r} 3x - 4y + 7z = 2 \\ 3z = 12 \end{array}\right\} .$$

which is a system in echelon form with no degenerate equations.

The only non-leading variable is y, so y can take any real value, say $y = t$. We have that $z = 4$, and $x = (2 + 4t - 28)/3 = -\frac{26}{3} + \frac{4}{3}t$. The intersection is thus:

$$\{\, (-\tfrac{26}{3} + \tfrac{4}{3}t, t, 4) : t \in \mathbb{R} \,\}.$$

This intersection is a line, having parametric equations:

$$\left.\begin{array}{l} x = -\frac{26}{3} + \frac{4}{3}\lambda \\ y = \qquad\quad\ \lambda \\ z = \ \ 4 \end{array}\right\}.$$

Feedback Question. [You should have shown all the steps in your calculations.]

(a)
$$\left.\begin{array}{r} -3x - \ \ y - 3z = \ \ 2 \\ 4x - 2y - 3z = -5 \end{array}\right\}.$$

To eliminate the x-term in the second equation, we add $\frac{4}{3}$ times the first equation to the second:

$$\left.\begin{array}{r} -3x - \quad y - 3z = \ \ 2 \\ -\frac{10}{3}y - 7z = -\frac{7}{3} \end{array}\right\}.$$

The system is now in echelon form, and contains no degenerate equation. We now apply back substitution.

The only non-leading variable is z, and so z can take any real value, say $z = t$. Then $y = (-\frac{7}{3}+7t)/(-\frac{10}{3}) = \frac{7}{10}-\frac{21}{10}t$. Now $-3x-(\frac{7}{10}-\frac{21}{10}t)-3t = 2$, and so $x = (2+\frac{7}{10}-\frac{21}{10}t+3t)/(-3) = (\frac{27}{10}+\frac{9}{10}t)/(-3)$, whence $x = -\frac{9}{10}-\frac{3}{10}t$. Thus, the solutions are $x = -\frac{9}{10} - \frac{3}{10}t$, $y = \frac{7}{10} - \frac{21}{10}t$, $z = t$, where t can be any real number.

(b)
$$\left.\begin{array}{r} 4y - 7z = \ \ 24 \\ 2x - 2y + \ \ z = -8 \\ x + 3y - 3z = \ \ 13 \\ -5x + 3y + 3z = \ \ \ 4 \end{array}\right\}.$$

We interchange the first two equations, so that the first equation has a non-zero x-term:

$$\left.\begin{array}{r} 2x - 2y + \ \ z = -8 \\ 4y - 7z = \ \ 24 \\ x + 3y - 3z = \ \ 13 \\ -5x + 3y + 3z = \ \ \ 4 \end{array}\right\}.$$

To eliminate the x-terms in all equations after the first, we add $-\frac{1}{2}$ times the first equation to the third, and $\frac{5}{2}$ times the first to the fourth (and leave the second equation alone) to get:

$$\left.\begin{array}{r} 2x - 2y + \quad z = \ \ -8 \\ 4y - \ \ 7z = \ \ \ 24 \\ 4y - \ \ \frac{7}{2}z = \ \ \ 17 \\ -2y + \frac{11}{2}z = -16 \end{array}\right\}.$$

To eliminate the y-terms in all equations after the second, we add -1 times the second equation the third and $\frac{1}{2}$ times the second equation to the fourth:

$$\left.\begin{array}{rl} 2x - 2y + z = & -8 \\ 4y - 7z = & 24 \\ \frac{7}{2}z = & -7 \\ 2z = & -4 \end{array}\right\}.$$

To eliminate the z-term in the fourth equation, we add $-(2)/(\frac{7}{2}) = -\frac{4}{7}$ times the third equation to the fourth:

$$\left.\begin{array}{rl} 2x - 2y + z = & -8 \\ 4y - 7z = & 24 \\ \frac{7}{2}z = & -7 \\ 0 = & 0 \end{array}\right\}.$$

We throw away the degenerate equation $0 = 0$, to obtain:

$$\left.\begin{array}{rl} 2x - 2y + z = & -8 \\ 4y - 7z = & 24 \\ \frac{7}{2}z = & -7 \end{array}\right\}.$$

The system is now in echelon form, and contains no degenerate equations. We shall solve this system by back substitution. All the variables are leading variables, and thus there is a unique solution. We have $z = -2$, $4y + 14 = 24$, so that $y = \frac{5}{2}$, and $2x - 5 - 2 = -8$, so that $x = -\frac{1}{2}$. Therefore, the only solution to the system is $x = -\frac{1}{2}$, $y = \frac{5}{2}$, $z = -2$.

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