MTH4103 (2013–14)

Geometry I

Solutions 3

29th January 2014

Practice Question 1. [It is extremely important that you do not mix up what is a vector and what is a scalar in an expression involving scalar products.]

(a) $\mathbf{a}\cdot\mathbf{b} = 6 + 2 - 28 = -20$.

(b) $\cos\theta = \mathbf{a}\cdot\mathbf{b}/(|\mathbf{a}||\mathbf{b}|) = -20/(\sqrt{21}\sqrt{62}) = -20/\sqrt{1302}$.

(c) $(\mathbf{a}\cdot(3\mathbf{a}+\mathbf{b}))\mathbf{b} = \left(\begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}\cdot\begin{pmatrix} 9 \\ -5 \\ -5 \end{pmatrix}\right)\mathbf{b} = 43\begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix} = \begin{pmatrix} 129 \\ -86 \\ 301 \end{pmatrix}$.

Practice Question 2. [Note how we must handle separately the case $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.]
We first handle the case when $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$. In this case, $|\mathbf{u}| = 0$ or $|\mathbf{v}| = 0$, and also $\mathbf{u}\cdot\mathbf{v} = 0$ (by definition), and so we have

$$|\mathbf{u}\cdot\mathbf{v}| = 0 = |\mathbf{u}||\mathbf{v}|.$$

Now suppose that $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, so that we can talk about the angle θ between them, and have by definition that $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$. Then we have

$$|\mathbf{u}\cdot\mathbf{v}| = |(|\mathbf{u}||\mathbf{v}|\cos\theta)| = |\mathbf{u}||\mathbf{v}||\cos\theta| \leqslant |\mathbf{u}||\mathbf{v}|,$$

since $|\mathbf{u}||\mathbf{v}| > 0$ and $|\cos\theta| \leqslant 1$ no matter what the angle θ is.

Practice Question 3. [These proofs are similar to some in your Week 3 lecture notes.]

(a) Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, and $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$. Then we have:

$$\begin{aligned}(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} &= \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{pmatrix}\cdot\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \\ &= (u_1+v_1)w_1 + (u_2+v_2)w_2 + (u_3+v_3)w_3 \\ &= u_1w_1 + v_1w_1 + u_2w_2 + v_2w_2 + u_3w_3 + v_3w_3 \\ &= (u_1w_1 + u_2w_2 + u_3w_3) + (v_1w_1 + v_2w_2 + v_3w_3) \\ &= \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}.\end{aligned}$$

(b) Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, and let α be a scalar. Then we have:

$$\begin{aligned}(\alpha\mathbf{u})\cdot\mathbf{v} = \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \\ \alpha u_3 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} &= (\alpha u_1)v_1 + (\alpha u_2)v_2 + (\alpha u_3)v_3 \\ &= \alpha(u_1v_1) + \alpha(u_2v_2) + \alpha(u_3v_3) \\ &= \alpha(u_1v_1 + u_2v_2 + u_3v_3) \\ &= \alpha(\mathbf{u}\cdot\mathbf{v}).\end{aligned}$$

Feedback Question. [Some appropriate calculations and explanations should have been given. Note that any nonzero scalar multiple of the given equation for the plane is correct.]

(a) $\begin{pmatrix} 1 \\ -3 \\ -7 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 5 \\ -6 \end{pmatrix} = (1)(-2)+(-3)(5)+(-7)(-6) = 25$, and so a Cartesian equation for Π is

$$-2x + 5y - 6z = 25.$$

(b) The point $(6, 1, 3)$ is *not on* Π since

$$-2(6) + 5(1) - 6(3) = -25,$$

and so the Cartesian equation for Π is not satisfied.

(c) The point $(-6, -1, -3)$ is *on* Π since

$$-2(-6) + 5(-1) - 6(-3) = 25,$$

and so the Cartesian equation for Π is satisfied.

Alternative solutions to Parts (b) and (c) are:

$$\begin{pmatrix} 6 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 5 \\ -6 \end{pmatrix} = -25 \neq 25 \quad \text{and} \quad \begin{pmatrix} -6 \\ -1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 5 \\ -6 \end{pmatrix} = 25.$$

(d) This distance is

$$\left| \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 5 \\ -6 \end{pmatrix} - 25 \right| \Big/ \left| \begin{pmatrix} -2 \\ 5 \\ -6 \end{pmatrix} \right| = |-29 - 25|/\sqrt{65} = 54/\sqrt{65}.$$

(e) This distance is

$$\left| \begin{pmatrix} 0 \\ -1 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 5 \\ -6 \end{pmatrix} - 25 \right| \Bigg/ \left| \begin{pmatrix} -2 \\ 5 \\ -6 \end{pmatrix} \right| = |25 - 25|/\sqrt{65} = 0.$$

Alternatively, we have $-2(0) + 5(-1) - 6(-5) = 25$, and so $(0, -1, -5)$ is on Π. Therefore the distance is 0.

(f) [It is perfectly acceptable to use the vector equation of Π here, except when a Cartesian equation is specifically called for.]

The general form of a Cartesian equation for Π is $ax + by + cz = d$, where $a, b, c, d \in \mathbb{R}$ and $(a, b, c) \neq (0, 0, 0)$. Now we suppose there is a vector $\mathbf{u}$ such that the points $U = (u_1, u_2, u_3)$ and $U' = (-u_1, -u_2, -u_3)$ with position vectors $\mathbf{u}$ and $-\mathbf{u}$ respectively are both on Π. Then

$$\begin{aligned} d &= a(-u_1) + b(-u_2) + c(-u_3) && \text{(as } U' \text{ is on } \Pi) \\ &= -(au_1 + bu_2 + cu_3) && \text{(by standard properties of } \mathbb{R}) \\ &= -d && \text{(as } U \text{ is on } \Pi). \end{aligned}$$

Thus $d = -d$ and so $d = 0$. Therefore, the most general form of a Cartesian equation for Π is

$$ax + by + cx = 0,$$

where $(a, b, c) \neq (0, 0, 0)$. (Actually, we have only shown that the condition $d = 0$ is necessary for Π to have the required property. We have not shown this condition is sufficient. But the working below, together with the observation that Π does have a point on it [such as $(0, 0, 0)$] shows this.) So now if (u_1, u_2, u_3) is on Π we get $a(-u_1) + b(-u_2) + c(-u_3) = -(au_1 + bu_2 + cu_3) = -0 = 0$, showing that $(-u_1, -u_2, -u_3)$ is on Π.

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