MTH4103 (2013–14)

Geometry I

Solutions 2 — 22nd January 2014

Practice Question 1. [Note: Draw a diagram to help you think.]

(a) Since P is the mid-point of AB, by Theorem 1.11 in the lecture notes

$$\mathbf{p} = (1 - \tfrac{1}{2})\mathbf{a} + \tfrac{1}{2}\mathbf{b} = \tfrac{1}{2}(\mathbf{a} + \mathbf{b}).$$

Similarly, $\mathbf{q} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$, $\mathbf{r} = \frac{1}{2}(\mathbf{c} + \mathbf{d})$ and $\mathbf{s} = \frac{1}{2}(\mathbf{d} + \mathbf{a})$.

(b) $\overrightarrow{PQ}$ represents the vector $\mathbf{q} - \mathbf{p}$, and from (a),

$$\mathbf{q} - \mathbf{p} = \tfrac{1}{2}(\mathbf{b} + \mathbf{c}) - \tfrac{1}{2}(\mathbf{a} + \mathbf{b}) = \tfrac{1}{2}(\mathbf{c} - \mathbf{a}).$$

$\overrightarrow{SR}$ represents the vector $\mathbf{r} - \mathbf{s}$, and from (a),

$$\mathbf{r} - \mathbf{s} = \tfrac{1}{2}(\mathbf{c} + \mathbf{d}) - \tfrac{1}{2}(\mathbf{d} + \mathbf{a}) = \tfrac{1}{2}(\mathbf{c} - \mathbf{a}).$$

(c) Since $\overrightarrow{PQ}$ and $\overrightarrow{SR}$ represent the same vector, it follows (from the definition of a parallelogram) that $PQRS$ is indeed a parallelogram.

Practice Question 2. (a) $3\mathbf{a} - 5\mathbf{b} = \begin{pmatrix} 3 \\ -3 \\ -12 \end{pmatrix} - \begin{pmatrix} 10 \\ -15 \\ 10 \end{pmatrix} = \begin{pmatrix} -7 \\ 12 \\ -22 \end{pmatrix}$.

(b) $|\mathbf{a} + 2\mathbf{b}| = \left| \begin{pmatrix} 5 \\ -7 \\ 0 \end{pmatrix} \right| = \sqrt{5^2 + (-7)^2 + 0^2} = \sqrt{25 + 49 + 0} = \sqrt{74}.$

(c) $|\mathbf{a}| = \sqrt{1^2 + (-1)^2 + (-4)^2} = \sqrt{18} = 3\sqrt{2}$, and the vector of length 1 in the same direction as $\mathbf{a}$ is

$$\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{1}{3\sqrt{2}}\mathbf{a} = \frac{\sqrt{2}}{6}\mathbf{a} = \begin{pmatrix} \sqrt{2}/6 \\ -\sqrt{2}/6 \\ -4\sqrt{2}/6 \end{pmatrix} = \begin{pmatrix} \sqrt{2}/6 \\ -\sqrt{2}/6 \\ -2\sqrt{2}/3 \end{pmatrix}.$$

[In expressions involving surds, I try to get all the surds in the numerator when this is looks neater, even if some of the fractions then have ‘common factors’ in their numerators and denominators. For example, we can divide the numerator and denominator of $\sqrt{2}/6$ by $\sqrt{2}$ to obtain $1/(3\sqrt{2})$.]

(d) The vector of length 1 in the opposite direction to $\mathbf{a}$ is

$$-\hat{\mathbf{a}} = \begin{pmatrix} -\sqrt{2}/6 \\ \sqrt{2}/6 \\ 2\sqrt{2}/3 \end{pmatrix}.$$

(e) The vector of length 3 in the same direction as $\mathbf{b}$ is

$$3\hat{\mathbf{b}} = \frac{3}{|\mathbf{b}|}\mathbf{b} = \frac{3}{\sqrt{17}}\mathbf{b} = \begin{pmatrix} 6/\sqrt{17} \\ -9/\sqrt{17} \\ 6/\sqrt{17} \end{pmatrix}.$$

Practice Question 3. (a) [Note that if your equations (especially the vector and parametric ones) differ from those below then you have not used the formulae given in the lectures. However, they may still be correct.]

Let $\mathbf{u} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$, and let $\mathbf{p}$ be the position vector of the point $(3, -1, 3)$, which we shall call P. Then a vector equation for the line through P in the direction of $\mathbf{u}$ is $\mathbf{r} = \mathbf{p} + \lambda\mathbf{u}$; that is,

$$\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}.$$

This gives the parametric equations:

$$\left.\begin{aligned} x &= 3 - \lambda \\ y &= -1 \\ z &= 3 + 4\lambda \end{aligned}\right\}.$$

Eliminating λ in the first and third equations above gives the Cartesian equations:

$$\frac{x-3}{-1} = \frac{z-3}{4},\ y = -1.$$

[Note that there was no λ to eliminate in the second equation, but we still need to include $y = -1$ in the Cartesian equations.]

(b) The point $(-1, -1, -7)$ is not on this line, since it does not satisfy, for example, the parametric equations. If it did, then we would have, from the first parametric equation, $-1 = 3 - \lambda$, and so $\lambda = 4$. But then $-7 \neq 3 + 4\lambda = 19$, and so the third parametric equation would not be satisfied.

(c) Yes. Plugging $\lambda = 1$ into the equations in Part (a) shows that $(2, -1, 7)$ is on this line.

Practice Question 4. (a) $\mathbf{a} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$.

The vector represented by $\overrightarrow{AB}$ is $\mathbf{b} - \mathbf{a} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$.

(b) [NB: Your equations may differ from those below and still be correct. You can check that the equations have the correct form and are satisfied by two distinct points on the line.]

The line ℓ through the points A and B is the line through A in the direction of $\mathbf{b} - \mathbf{a}$, and so a vector equation for ℓ is $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$; that is,

$$\mathbf{r} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}.$$

This gives the parametric equations:

$$\left.\begin{aligned} x &= -2 + 4\lambda \\ y &= 3 + \lambda \\ z &= 5 + 2\lambda \end{aligned}\right\}.$$

Eliminating λ gives the Cartesian equations:

$$\frac{x+2}{4} = \frac{y-3}{1} = \frac{z-5}{2}.$$

(c) [Note: Draw a diagram to help you think.]

Let O be the origin, so $\overrightarrow{OB}$ represents the position vector $\mathbf{b}$ of B. Now suppose $\overrightarrow{BC}$ represents $\mathbf{b} - \mathbf{a}$. Then by the Triangle Rule, $\overrightarrow{OC}$ represents $\mathbf{c}$ where

$$\mathbf{c} = \mathbf{b} + (\mathbf{b} - \mathbf{a}) = 2\mathbf{b} - \mathbf{a} = \begin{pmatrix} 6 \\ 5 \\ 9 \end{pmatrix}.$$

This is the position vector of the point C, and so $C = (6, 5, 9)$.

The point C is on the line ℓ, since B is on ℓ, and $\overrightarrow{BC}$ is in the direction of ℓ. Note that we did not need to determine C in order to show it lies on ℓ. Furthermore, we have $\mathbf{b} = \mathbf{a} + (\mathbf{b} - \mathbf{a})$, and so $\mathbf{c} = \mathbf{a} + 2(\mathbf{b} - \mathbf{a})$, and so we see that the value $\lambda = 2$ in Part (b) gives rise to the point C.

Alternatively, we can check that equations for ℓ are satisfied. For example, $C = (6, 5, 9)$ is on ℓ because our Cartesian equations for ℓ are satisfied:

$$\frac{6+2}{4} = \frac{5-3}{1} = \frac{9-5}{2} \quad \text{(which are all equal to 2).}$$

Feedback Question. (a) We say that $\mathbf{a}$ and $\mathbf{b}$ are *orthogonal* if $\mathbf{a}\cdot\mathbf{b} = 0$.

(b) Any nonzero vector $\mathbf{b}$ with $\mathbf{a}\cdot\mathbf{b} = 0$ will do. One such $\mathbf{b}$ is $\begin{pmatrix} 13 \\ 1 \\ 2 \end{pmatrix}$, for $\begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 13 \\ 1 \\ 2 \end{pmatrix} = -13 + 3 + 10 = 0$. (There are simpler possibilities for $\mathbf{b}$, such as the position vector of the point $(3, 1, 0)$. Think about how I obtained the second answer.)

(c) [Many people approach this (sort of) question by assuming that $\mathbf{a} = \mathbf{0}$ (the result you are trying to prove), and then deduce the $\mathbf{a}\cdot\mathbf{b} = 0$. Please do not do this; you will score no marks in examinations or tests if you do, because such an approach is wrong.]

Suppose that $\mathbf{a}\cdot\mathbf{b} = 0$ for every vector $\mathbf{b}$. Then, in particular, if $\mathbf{b} = \mathbf{a}$, we must have $0 = \mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$, so we must have $|\mathbf{a}| = 0$, and so $\mathbf{a} = \mathbf{0}$.

Here is an alternative proof that uses coördinates.

Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, and suppose that $\mathbf{a}\cdot\mathbf{b} = 0$ for every vector $\mathbf{b}$. Then

$$0 = \mathbf{a}\cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = a_1 + 0 + 0 = a_1,$$

$$0 = \mathbf{a}\cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 + a_2 + 0 = a_2,$$

$$0 = \mathbf{a}\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 + 0 + a_3 = a_3,$$

and so $\mathbf{a} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}$.

(d) If we take $\mathbf{b} = \mathbf{0}$ then $\mathbf{a}\cdot\mathbf{b} = 0 \neq 1$. (The proof really is that short.) There is also no vector $\mathbf{a}$ with the property that $\mathbf{a}\cdot\mathbf{b} = 1$ for all nonzero vectors $\mathbf{b}$, for if $\mathbf{c}$ is a nonzero vector such that $\mathbf{a}\cdot\mathbf{c} = 1$ then $\mathbf{b} = 2\mathbf{c} \neq \mathbf{0}$ and $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot(2\mathbf{c}) = 2(\mathbf{a}\cdot\mathbf{c}) = 2 \neq 1$.

Dr John N. Bray, 22nd January 2014