MTH4103 (2013–14)

Geometry I

Solutions 1 **15^{th} January 2014**

Practice Question 1. Let $\overrightarrow{AB}$ represent $\mathbf{u}$. As $\overrightarrow{BB}$ represents $\mathbf{0}$, we see by the Triangle Rule that $\overrightarrow{AB}$ represents $\mathbf{u}+\mathbf{0}$, and so $\mathbf{u}+\mathbf{0}=\mathbf{u}$.

Practice Question 2. We have

$$\begin{aligned}
((-\mathbf{u})+\mathbf{v})+\mathbf{u} &= \mathbf{u}+((-\mathbf{u})+\mathbf{v}) && \text{by Theorem 1.2 (Commutative Law)}\\
&= (\mathbf{u}+(-\mathbf{u}))+\mathbf{v} && \text{by Theorem 1.4 (Associative Law)}\\
&= \mathbf{0}+\mathbf{v} && \text{by Theorem 1.6 (Inverse Law)}\\
&= \mathbf{v}+\mathbf{0} && \text{by Theorem 1.2 (Commutative Law)}\\
&= \mathbf{v}. && \text{by Theorem 1.5 (Identity Law)}
\end{aligned}$$

In this question, the words Commutative, Associative, Identity and Inverse all refer to the operation of vector addition. (Note that there are other routes to proving this result, so that if your answer differs from mine, it does not necessarily mean it is wrong. This happens quite a lot in Mathematics.)

Practice Question 3. We make use of the facts that $(-1)\mathbf{v}=-\mathbf{v}$ and $\alpha(\beta\mathbf{v})=(\alpha\beta)\mathbf{v}$ for all scalars α, β and all vectors $\mathbf{v}$. We have:

$$-(\lambda\mathbf{w})=(-1)(\lambda\mathbf{w})=((-1)\lambda)\mathbf{w}=(-\lambda)\mathbf{w},$$

and

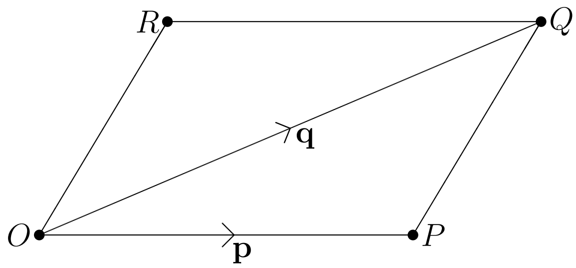
$$(-\lambda)\mathbf{w}=((-1)\lambda)\mathbf{w}=(\lambda(-1))\mathbf{w}=\lambda((-1)\mathbf{w})=\lambda(-\mathbf{w}).$$

Feedback Question. Parts (a) and (b) are taken directly from your lecture notes. You should also understand the difference between a point P, the bound vector $\overrightarrow{OP}$, and the free vector $\mathbf{p}$ represented by this bound vector. You should not, for example, write $\mathbf{p}=\overrightarrow{OP}$ when you mean that $\mathbf{p}$ is represented by $\overrightarrow{OP}$. You should also be able to give correct mathematical definitions by appropriately modifying definitions that are in your lecture notes.

(a) The figure $OPQR$ is a *parallelogram* if $\overrightarrow{OP}$ and $\overrightarrow{RQ}$ represent the same vector. (This is equivalent to $\overrightarrow{OR}$ and $\overrightarrow{PQ}$ representing the same vector, something I dubbed the *Parallelogram Axiom*.)

(b) The *position vector* $\mathbf{p}$ of a point P is the (free) vector represented by the bound vector $\overrightarrow{OP}$.

(c) Many of you did this question wrong, as you assumed that $\mathbf{p}$ and $\mathbf{q}$ must be represented by the sides of parallelogram. However, if we are more careful, we note that the parallelogram is $OPQR$, and so $\overrightarrow{OQ}$ is a *diagonal* of the parallelogram. Drawing a diagram helps. The initial diagram is as follows.



(i) $\overrightarrow{OP}$ represents the position vector $\mathbf{p}$ of P, so the required expression is simply $\mathbf{p}$.

(ii) $\overrightarrow{OQ}$ represents the position vector $\mathbf{q}$ of Q, and so $\overrightarrow{QO}$ represents $-\mathbf{q}$.

(iii) Let $\mathbf{r}$ be the position vector of R, that is, the vector represented by $\overrightarrow{OR}$. Then, by the Parallelogram Rule, we have $\mathbf{q} = \mathbf{p} + \mathbf{r}$, and so $\mathbf{r} = \mathbf{q} - \mathbf{p}$, that is, $\overrightarrow{OR}$ represents $\mathbf{q} - \mathbf{p}$. (Here, I allow the 'obvious' deduction of $\mathbf{r} = \mathbf{q} - \mathbf{p}$ from $\mathbf{q} = \mathbf{p} + \mathbf{r}$ to be taken as read. To prove this, one can add $-\mathbf{p}$ to both sides of $\mathbf{q} = \mathbf{p} + \mathbf{r}$ to get

$$\begin{aligned}\mathbf{q} - \mathbf{p} = (\mathbf{p} + \mathbf{r}) - \mathbf{p} = (\mathbf{r} + \mathbf{p}) - \mathbf{p} &= (\mathbf{r} + \mathbf{p}) + (-\mathbf{p}) \\ &= \mathbf{r} + (\mathbf{p} + (-\mathbf{p})) = \mathbf{r} + \mathbf{0} = \mathbf{r},\end{aligned}$$

which uses various theorems/definitions established in the lectures.)

(iv) By the Triangle Rule, we have that $\overrightarrow{PQ}$ represents $-\mathbf{p} + \mathbf{q} = \mathbf{q} - \mathbf{p}$. Alternatively, since $OPQR$ is a parallelogram, $\overrightarrow{PQ}$ represents the same vector as $\overrightarrow{OR}$, and in (iii) this vector was shown to be $\mathbf{q} - \mathbf{p}$.

(v) Since $OPQR$ is a parallelogram, $\overrightarrow{RQ}$ represents the same vector $\mathbf{p}$ as $\overrightarrow{OP}$. Thus $\overrightarrow{QR}$ represents $-\mathbf{p}$.

(vi) Again, let $\mathbf{r}$ be the vector represented by $\overrightarrow{OR}$. By the Triangle Rule, $\overrightarrow{RP}$ represents

$$-\mathbf{r} + \mathbf{p} = -(\mathbf{q} - \mathbf{p}) + \mathbf{p} = -\mathbf{q} + \mathbf{p} + \mathbf{p} = 2\mathbf{p} - \mathbf{q}.$$

Dr John N. Bray, 15th January 2014