MTH4103 (2013–14)

Geometry I

Coursework 10

Not for handing in

There is no Feedback Question in this final Geometry I coursework, but you should still attend your Week 12 Exercise Class (on 26th March 2014), having attempted to do the Practice Questions beforehand.
Even though there is nothing to hand in, you should still show clearly all the steps in your calculations in all the questions that you attempt.

Practice Question 1. Let t be a linear transformation of $\mathbb{R}^n$ represented by the $n \times n$ matrix A. Define what is meant by an *eigenvector* $\mathbf{v}$ of t, and of A, and what is meant by the *eigenvalue* of t, and of A, corresponding to $\mathbf{v}$.

Practice Question 2. For each of the following matrices A, determine the characteristic polynomial of A, all the eigenvalues of A, and for each eigenvalue λ of A, determine the set of all eigenvectors of A with corresponding eigenvalue λ:
(a) $\begin{pmatrix} 1 & -1 \\ 3 & 5 \end{pmatrix}$; (b) $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$; (c) $\begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix}$.

Practice Question 3. Let t be the linear transformation of $\mathbb{R}^3$ represented by

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 4 \end{pmatrix}.$$

(a) Determine the characteristic polynomial and all the eigenvalues of A.

(b) Determine one eigenvector of A with corresponding eigenvalue -1.

(c) Determine a vector equation for a line ℓ through the origin, such that ℓ is fixed by t, and for all points P on ℓ, if $\mathbf{x}$ is the position vector of P then $t(\mathbf{x}) = -\mathbf{x}$.

Practice Question 4. Let A be an $n \times n$ matrix and suppose that $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of A, with both having the same corresponding eigenvalue λ. Prove that if $\mathbf{u} + \mathbf{v} \neq \mathbf{0}_n$ then $\mathbf{u} + \mathbf{v}$ is also an eigenvector of A with corresponding eigenvalue λ.

Practice Question 5. Recall that R_θ is the matrix representing rotation about the origin (in the (x, y)-plane) through an angle θ, and that S_θ is the matrix representing a reflexion (in the (x, y)-plane) in the line through the origin at angle θ from the x-axis.

(a) Compute $A = S_\theta S_0$ and $B = S_\pi S_\theta$, and show that both A and B represent rotations.

(b) Prove that $(S_\theta)^{-1} = S_\theta$.

(c) Prove, without using matrix entries, that for all angles $\theta_1, \theta_2, \theta_3$, we have:

$$(S_{\theta_1} S_{\theta_2} S_{\theta_3})^{-1} = S_{\theta_3} S_{\theta_2} S_{\theta_1}.$$

Practice Question 6. Write down matrices R_1 and R_2 representing a rotation through angle $\pi/2$ about the x-axis in $\mathbb{R}^3$ and a rotation through angle $\pi/2$ about the z-axis in $\mathbb{R}^3$.

(a) Compute the matrix product $R_3 = R_2 R_1$.

(b) Show that 1 is an eigenvalue of R_3 and compute an eigenvector corresponding to this eigenvalue.

(c) R_3 is a rotation. What is the direction of its fixed axis?

(d) Show that $(R_3)^3 = \mathrm{I}_3$. Deduce that R_3 represents a rotation though angle $\pm 2\pi/3$ about its fixed axis. *(Whether the sign is + or − depends on which direction you assign to the axis of rotation.)*

(e) Does $R_1 R_2$ have the same fixed axis as $R_2 R_1$? Is $R_1 R_2$ also a rotation through $\pm 2\pi/3$?

Dr John N. Bray, 17th March 2014