MTH4103 (2013–14)

Geometry I

Coursework 8 — To hand in on 12th March 2014

A solution to the Feedback Question, stapled if on more than one sheet, and with your full name (last name underlined) and student number, should be handed in to your tutor first thing in your Week 10 Exercise Class.
By the beginning of your Week 10 Exercise Class (on 12th March 2014), you should already have tried the Practice Questions, on which you can ask for help in the Exercise Class. You will not receive any help on your Feedback Question. Show clearly all the steps in your calculations.

Practice Question 1. For each of the following matrices, calculate its determinant, say whether the matrix is invertible, and if it is, find its inverse.

$$\text{(a)} \quad \begin{pmatrix} 7 & -4 \\ 5 & -3 \end{pmatrix}; \qquad \text{(b)} \quad \begin{pmatrix} 6 & -4 \\ -15 & 10 \end{pmatrix}; \qquad \text{(c)} \quad \begin{pmatrix} 4 & -3 \\ 3 & 2 \end{pmatrix}.$$

Practice Question 2. Calculate the determinant of each of the following matrices.

$$\text{(a)} \quad \begin{pmatrix} 1 & -3 & 2 \\ -1 & 1 & -1 \\ -4 & -2 & -1 \end{pmatrix} \qquad \text{and} \qquad \text{(b)} \quad \begin{pmatrix} 4 & 2 & 1 \\ 2 & 3 & 1 \\ 0 & 3 & -1 \end{pmatrix}.$$

Practice Question 3. Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Show that $\det A = 1$ and, using the formula in your notes, find the inverse of A. Hence find all solutions x_1, x_2, x_3 of the simultaneous equations

$$\left.\begin{aligned} 2x_1 + x_2 + x_3 &= y_1 \\ x_1 + 2x_2 + x_3 &= y_2 \\ x_1 + x_2 + x_3 &= y_3 \end{aligned}\right\} .$$

Practice Question 4. Prove that if A and B are invertible $n \times n$ matrices then $(AB)^{-\mathsf{T}} = A^{-\mathsf{T}} B^{-\mathsf{T}}$. Recall that $A^{-\mathsf{T}}$ means $(A^{-1})^{\mathsf{T}}$, which is equal to $(A^{\mathsf{T}})^{-1}$. You may use results proved in lectures.

Practice Question 5. Prove for all $m \times n$ matrices A and B and all scalars λ that

(a) $(A^\mathsf{T})^\mathsf{T} = A$;

(b) $(A + B)^\mathsf{T} = A^\mathsf{T} + B^\mathsf{T}$;

(c) $(-A)^\mathsf{T} = -(A^\mathsf{T})$; and

(d) $(\lambda A)^\mathsf{T} = \lambda(A^\mathsf{T})$.

(Thus the notations $-A^\mathsf{T}$ and λA^T are unambiguous.)

Practice Question 6. Prove that $\operatorname{adj}(A+B) = \operatorname{adj} A + \operatorname{adj} B$ for all 2×2 matrices A and B, where $\operatorname{adj} A$ denotes the adjugate of A. Show that this equality is not universally true for 1×1 or 3×3 matrices. (It is actually always false in the 1×1 case, and seldom true in the 3×3 case.)

Feedback Question. For each of the following statements, say whether it is true or false, and justify your answer with a proof (if it is true) or a counterexample (if it is false).

(a) Whenever A is an invertible $n \times n$ matrix over $\mathbb{R}$ then $-2A$ is also invertible.

(b) Whenever A and B are invertible 2×2 matrices then $A+B$ is also invertible.

(c) For each invertible 2×2 matrix A, we have $\det A \neq 0$ and $\det(A^{-1}) = 1/\det A$.

(d) For all scalars α and all 3×3 matrices A, we have $\det(\alpha A) = \alpha^3 \det A$.

(e) For all scalars α and all 2×2 matrices A, we have $\det(\alpha A) \neq \alpha \det A$.

Dr John N. Bray, 3rd March 2014