MTH4103 (2013–14)

Geometry I

Coursework 6 **To hand in on 26th February 2014**

A solution to the Feedback Question, stapled if on more than one sheet, and with your full name (last name underlined) and student number, should be handed in to your tutor first thing in your Week 8 Exercise Class.
By the beginning of your Week 8 Exercise Class (on 26th February 2014), you should already have tried the Practice Questions, on which you can ask for help in the Exercise Class. You will not receive any help on your Feedback Question.
Show clearly all the steps in your calculations.

Practice Question 1. We proved in lectures that for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, we have

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) \quad \text{and} \quad (\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w}).$$

Apply these facts to give a short proof, without using coördinates, that for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, we have

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}).$$

Practice Question 2. Consider the parallelogram $ABCD$, with $A = (1, -3, 1)$, $B = (1, -1, 4)$ and $C = (4, 1, 0)$. Determine the area of $ABCD$.

Practice Question 3. Let $ABCD$ be a parallelogram, and let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ be the position vectors of A, B, C and D respectively. In lectures I showed that that parallelogram $ABCD$ has area $|(\mathbf{b} - \mathbf{a}) \times (\mathbf{d} - \mathbf{a})|$. Determine the area of parallelogram $ABCD$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ alone.

Practice Question 4. Find the distance from the point $(-2, 1, 5)$ to the line defined by the Cartesian equations

$$\frac{x-4}{3} = \frac{y+2}{3} = \frac{z-1}{-5}.$$

Practice Question 5. (a) Find the distance between the lines defined by

$$\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -6 \\ 9 \\ -3 \end{pmatrix}.$$

(b) Find the distance between the lines defined by

$$\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}.$$

Practice Question 6. Prove that $|\mathbf{a} \times \mathbf{b}| = \sqrt{|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2}$ for all vectors $\mathbf{a}$ and $\mathbf{b}$.

Practice Question 7.

(a) Given that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$ for all vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, prove that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$ for all vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

(b) The set $\mathbb{H}$ of *quaternions* is defined to be the set of ordered pairs $(\alpha, \mathbf{a})$, where $\alpha \in \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^3$. (The notation $\mathbb{H}$ is used, since the quaternions were discovered by William Rowan Hamilton (1805–1865).) Quaternion addition and multiplication are defined by $(\alpha, \mathbf{a}) + (\beta, \mathbf{b}) = (\alpha + \beta, \mathbf{a} + \mathbf{b})$ and $(\alpha, \mathbf{a}).(\beta, \mathbf{b}) = (\alpha\beta - \mathbf{a}\cdot\mathbf{b}, \alpha\mathbf{b} + \beta\mathbf{a} + \mathbf{a} \times \mathbf{b})$. It is quite easy to see that quaternion addition is associative and commutative, with additive identity $(0, \mathbf{0})$, and additive inverse $-(\alpha, \mathbf{a}) = (-\alpha, -\mathbf{a})$. Prove that quaternion multiplication is associative, but not commutative. (One can show that the multiplicative identity for quaternions is $(1, \mathbf{0})$, and that $(\alpha, \mathbf{a}) \neq (0, \mathbf{0})$ has two-sided multiplicative inverse $(\lambda\alpha, -\lambda\mathbf{a})$, where $\lambda = 1/(\alpha^2 + |\mathbf{a}|^2)$.)

Note that those of you doing MTH4104: Introduction to Algebra will have met some different-looking quaternions on Coursework 3. Despite appearances to the contrary, those quaternions are the "same" as the ones above, in a sense that mathematicians can make precise.

Feedback Question.

(a) Determine a Cartesian equation for the plane through the points $(2, 5, 1)$, $(-5, 2, -1)$ and $(1, -2, -4)$.

(b) Adapt the method used in (a) to determine a Cartesian equation for the plane through the point $(-4, 2, 1)$ and parallel to the plane through the points $(3, -1, -2)$, $(4, 1, -3)$ and $(2, 3, -1)$.

Dr John N. Bray, 10th February 2014