Model Theory LTCC Exam (due 03 Dec 2018)

Please send me the solutions by email to `i.tomasic@qmul.ac.uk`. The solutions should be as self-contained as possible, and include precise references to any results used. Scanned manuscripts are acceptable (if legible).

1. [First-order logic] Let $(G, \cdot)$ be a group. An element $g \in G$ is called *torsion* if g is of finite order, i.e., there exist an n such that $g^n = 1$. A group G is *torsion* if every element is torsion. A group G is *torsion-free* if every element different from 1 is of infinite order. An abelian group $(A, +)$ is *divisible* if for every element $a \in A$ and every $n > 0$ there exists an $x \in A$ such that $nx = \underbrace{x + \cdots + x}_{n \text{ times}} = a$.

(a) Find a first-order axiomatisation for the class of torsion-free divisible abelian groups and prove that this theory is complete.
(b) Prove that the property 'G is torsion' cannot be axiomatised in a first-order way in the language of groups.

2. [Ordering $\mathbb{C}$] It is well-known in algebra that $\mathbb{C}$ cannot be made into an ordered field. Prove that it cannot even be made into a total order by a first-order formula in the language of rings.

3. [$\forall\exists$-sentences and Lefschetz principle]

(a) Let θ be an $\forall\exists$-sentence, i.e.,

$$\theta \equiv \forall x_1 \dots \forall x_n \exists y_1 \dots \exists y_m \, \varphi(x_1, \dots, x_n, y_1, \dots, y_m),$$

where φ is quantifier-free. Let M_i, $i \in I$ be a chain of structures indexed by a linear order $(I, <)$ such that $M_i \models \theta$ for all $i \in I$. Prove that

$$\bigcup_{i \in I} M_i \models \theta.$$

(b) Let $\sigma : \mathbb{C}^n \to \mathbb{C}^n$ be an algebraic automorphism of $\mathbb{C}^n$ viewed as an algebraic variety (the affine n-space, sometimes denoted $\mathbb{A}^n$). In other words, the components of σ are polynomial maps. Prove the following statement.
If $\sigma^2 = 1$, then σ has a fixed point.

4. [Morley rank in topological terms] Let M be an $\aleph_0$-saturated model of a complete theory T. There is a notion of *Cantor-Bendixson derivative* of a topological space, see: `http://en.wikipedia.org/wiki/Derived_set_(mathematics)`

We shall apply these considerations to the Stone space $X = S_n(M)$. Given a type $p \in X$, we say that its *Cantor-Bendixson rank* is α, written $\mathrm{CB}(p) = \alpha$, if $p \in X^\alpha \setminus X^{\alpha+1}$.

Prove the following comparison to the Morley rank:

(a) $\mathrm{MR}(p) \geq \mathrm{CB}(p)$;
(b) if T is totally transcendental, $\mathrm{MR}(p) = \mathrm{CB}(p)$.

Dr. Ivan Tomašić, QMUL