9.1

Discrete probability

Syllabus

Discrete probability, relevance to performance

9.2

Consider a situation where
we know something is going to happen,
but we don't know exactly what.
For example we are going to toss a coin,
but we don't know whether it will fall heads or tails.

Suppose we know that the outcome will be exactly one
of a set S of possible outcomes.

We call S the *sample space*, and its members are called *points*.

9.3

Example. Suppose we are going to toss a coin 5 times.
Each time it could fall heads or tails.
So the sample space is the set of ordered 5-tuples
from $\{\text{Heads, Tails}\}$,
i.e. it is

$$\{\text{Heads, Tails}\}^5.$$

This sample space has cardinality $2^5 = 32$.

9.4

An *event* is a subset of the sample space.

For example the event 'the coin comes down the same at each of the five tosses' is the event

$$\{\text{HHHHH, TTTTT}\}.$$

The empty set $\emptyset$ is the *impossible event*.

We can form the union $A \cup B$ and the intersection $A \cap B$ of two events,
and the complement $A' = S \setminus A$ of an event A.

9.5

A *probability space* (S, P) is a sample space S together with a function P whose domain is the set of events of S;
$P(A)$, called the *probability* of A, measures
how probable it is that the event A will occur,
i.e. that the outcome will be some member of A.

The function P must have the following properties.

(1) $0 \leqslant P(A)$ for all events A, and $P(S) = 1$.

This is because 1 means certainty and 0 means impossibility.

9.6

(2) If the events A and B are disjoint ($A \cap B = \emptyset$), then

$$P(A \cup B) = P(A) + P(B).$$

This generalises: if $A_1, \ldots, A_n$ are events that are disjoint from each other, then

$$P(A_1 \cup \ldots \cup A_n) = P(A_1) + \ldots + P(A_n).$$

A function P with domain the set of events, and satisfying (1) and (2), is called a *probability function*.

9.7

Suppose a is a point in the sample space.
Then $\{a\}$ is an event, viz. the event that a occurs,
so it has probability $P(\{a\})$.

We say that this number $P(\{a\})$ is the *probability* of the point a, and we write it $P(a)$.

The probability of an event $\{a, b, c, d\}$ is the sum of the probabilities of the points in it:

$$P(\{a, b, c, d\}) = P(a) + P(b) + P(c) + P(d).$$

9.8

Example

Suppose the sample space consists of n points,
all equally probable to occur.

Since the probabilities add up to 1, they must each be $\frac{1}{n}$.

So in this sample space the probability of an event A is always

$$\frac{|A|}{n}.$$

9.9

Example

Suppose a fair coin is tossed six times. What is the probability that it will be heads exactly three times?

ANSWER. The sample space S is the set of all ordered 6-tuples from $\{\text{Heads, Tails}\}$, for example HHHHHH, HTHTHT etc.

There are $2^6 = 64$ points in S, all equally probable, so each has probability $1/64$.

9.10

The number of ways of getting exactly three Hs is equal to the number of ways of choosing 3 elements from a set of six elements. (The six elements are ‘H in first place’, ‘H in second place’ etc.)

So the size of the event ‘exactly three heads’ is $C(6, 3)$, which is

$$\frac{6!}{3!(6-3)!} = \frac{6!}{3! \times 3!} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20.$$

9.11

So the probability of the event is 20 times $1/64$, in other words

$$\frac{20}{64} = \frac{5}{16},$$

which is about 0.313.

9.12

In a sample space S, we write $S \setminus A$ or A' for the event that A *doesn’t* occur.

FACT. The event A' has probability $1 - P(A)$.

This is because A and A' are disjoint, so

$$P(A) + P(A') = P(A \cup A') = P(S) = 1.$$

So

$$P(A') = 1 - P(A).$$

9.13

Example

A fair coin is tossed five times. What is the probability that at least once a Head will be followed by a Tail?

ANSWER. Let A be the event that at least once a head is followed by a tail.
Then A' is the event that all the tails (if any) come before all the heads.

9.14

So A' is the union of the events:

(a) For some i between 1 and 5 inclusive, there are i tails followed by $5 - i$ heads.

(b) Only heads occur.

Event (a) consists of 5 points, one for each i.
Event (b) consists of exactly one point.
So A' has size 6, and we have

$$P(A) \;=\; 1 - P(A') \;=\; 1 - \frac{6}{2^5} = \; 1 - \frac{6}{32} \;=\; 1 - \frac{3}{16} \;=\; \frac{13}{16}.$$

9.15

Example

We assume a coin is biased and the probability of getting a head on any toss is 2/3. What is the probability of getting exactly three heads if the coin is tossed six times?

ANSWER. The tosses are independent,
so the probability of getting three heads on the first three throws is equal to the probability of getting three heads in any other given positions.

9.16

The probability of getting Tails is $1 - 2/3 = 1/3$.
So the probability of getting heads the first three times and tails the last three is

$$\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \;=\; \frac{8}{729}.$$

So the probability of getting exactly three heads is

$$C(6,3) \times \frac{8}{729} \;=\; 20 \times \frac{8}{729} \;=\; \frac{160}{729},$$

about 0.219.

9.17

Suppose (S, P) is a probability space and C is an event with $P(A) > 0$.
Suppose we are given the information that the outcome from S is going to be in C.
This changes the probabilities of the events;
for example C now becomes a certainty.
We write $P(A|C)$ for the new probability of an event A,
and we call it the *probability of* A *given* C.

9.18

We have

$$P(A|C) = \frac{P(A \cap C)}{P(C)}.$$

Rearranging, this is equivalent to the important formula

$$P(A \cap C) = P(C) \times P(A|C).$$

9.19

We say that A is *independent of* C if

$$P(A) = P(A|C).$$

(I.e. the information that C will occur tells us nothing new about whether A will occur.)

Using the formula for $P(A|C)$, this equation says

$$P(A) = \frac{P(A \cap C)}{P(C)},$$

in other words A is independent of C if

$$P(A \cap C) = P(A) \times P(C).$$

9.20

The same calculation shows that C is independent of A if and only if

$$P(C \cap A) = P(C) \times P(A),$$

which is equivalent to $P(A \cap C) = P(A) \times P(C)$.

So A is independent of C if and only if
C is independent of A,
and in this case we say A and C are *independent*.

9.21

Example

A box of chocolates contains twelve identical-looking chocolates.
Eight of them are truffles and four are raspberry ripple.
If you choose three chocolates at random, what is the probability that they will all be truffles?

Important: This is not like tossing a coin.
When you have taken a truffle out of the box, the proportions have changed.
So the chances of a truffle on the second choice depend on whether you got a truffle at your first choice.

9.22

Write A_i for the event that the i-th choice is a truffle.

The probability of A_1 is $8/12 = 2/3$.
If A_1 occurred, the probability of A_2 is

$$P(A_2|A_1) = \frac{7}{11}.$$

If A_1 and A_2 occurred, the probability of A_3 is

$$P(A_3|A_1 \cap A_2) = \frac{6}{10} = \frac{3}{5}.$$

9.23

So

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3) &= P(A_1 \cap A_2) \times P(A_3|A_1 \cap A_2) \\ &= P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 \cap A_2) \\ &= \tfrac{2}{3} \times \tfrac{7}{11} \times \tfrac{3}{5} \\ &= \tfrac{14}{55}. \end{aligned}$$

This is about 0.255.

9.24

Example from Exam 2003.

A fair coin is tossed three times.
Let A be the event that precisely two heads occur.
Let B be the event that at least one tail occurs.
Calculate the probabilities $P(A)$ and $P(B)$,
and determine whether A and B are independent.

9.25

The probability of getting a head when we toss the coin is $1/2$.
Similarly for getting a tail.
B' is the event that we get HHH,
which has probability $(1/2)^3$.
So $P(B) = 1 - P(B') =$

$$1 - \left(\frac{1}{2}\right)^3 \;=\; 1 - \frac{1}{2^3} \;=\; 1 - \frac{1}{8} \;=\; \frac{7}{8}.$$

9.26

The event A of getting exactly two heads is the event that we have either HHT or HTH or THH.
Each of these possibilities has probability $(1/2)^3 = 1/8$,
and they are disjoint, so

$$P(A) \;= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \;=\; \frac{3}{8}.$$

9.27

To check whether A and B are independent,
we have to find whether

$$P(A \cap B) = P(A) \times P(B).$$

The right side of this equation is

$$\frac{7}{8} \times \frac{3}{8} \;=\; \frac{21}{64}.$$

9.28

The probability on the left is the same as the probability of A, since A implies B.
So

$$P(A \cap B) \;=\; \frac{3}{8} \;\neq\; \frac{21}{64} \;=\; P(A) \times P(B).$$

Hence the two events are not independent.

9.29

Example

Consider the events which might happen this evening:

A: My sister will play her pop music.
B: I will feel ill.
C: A tube problem will make me late home.

Suppose these events are independent, with probabilities

$$P(A) = 0.5, \quad P(B) = 0.2, \quad P(C) = 0.1.$$

Then we can complete a probability truth table for all eight combinations of these events:

9.30

	A	B	C	probability
1.	T	T	T	$0.5 \times 0.2 \times 0.1 = 0.01$
2.	T	T	F	$0.5 \times 0.2 \times (1 - 0.1) = 0.09$
3.	T	F	T	0.04 (similarly)
4.	T	F	F	0.36
5.	F	T	T	0.01
6.	F	T	F	0.09
7.	F	F	T	0.04
8.	F	F	F	0.36

Check: the numbers add up to 1.

9.31

Suppose I will get my homework done as long as there are no tube problems, and provided I don't both feel ill and have to compete with my sister's pop music.

Then I will get my homework done in cases 4, 6 and 8.

So the probability I will get my homework done is

$$0.36 + 0.09 + 0.36 \;=\; 0.81.$$

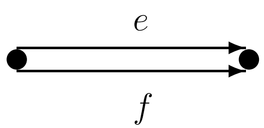
9.32

Suppose we have a network N.
(Assume each edge has capacity 1; this won't matter.)

Suppose each edge e has a certain probability $P(e)$ of breaking, and these probabilities are independent.
The *dependability* of the network is the probability that there will be an unbroken directed path from source to sink.

9.33

Example



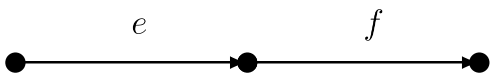
with probabilities of breaking $P(e) = 0.3$, $P(f) = 0.1$.
There is an unbroken path through if and only if not both of the edges e, f break.

So the dependability is

$$1 - P(e)P(f) \ = \ 1 - (0.3 \times 0.1) \ = \ 1 - 0.03 \ = \ 0.97.$$

9.34

Example



with the same probabilities $P(e) = 0.3$, $P(f) = 0.1$.
In this case there is an unbroken path if and only if both the edges e, f are unbroken.

So the dependability is

$$(1 - P(e))(1 - P(f)) \ = \ 0.7 \times 0.9 \ = \ 0.63.$$