Algorithms

Syllabus

> Recursion;
> functions defined by recursive relationships;
> loops.

Now we are going to open up the box that contains the function f.
We shall describe *algorithms* that say how a function is calculated.

Our descriptions will be pictures.
An oval represents input or output.
The input oval tells us what kind of object is input.
The output oval says what output is 'returned'.

A diamond represents a question or test.

A rectangle represents a simple calculation to be done. For example

$$x := 8 * x$$

represents the following calculation:

> Take the number called x. Multiply it by 8.
> (This is $8 \star x$, where we use $\star$ instead of $\times$ to avoid confusion with x). Then move the name x to the number you have just calculated. ('$x :=$' means this.)

Muhammad Ibn Musa

Al-Khowarizmi

'Mr Algorithm'

Iran, AD 800
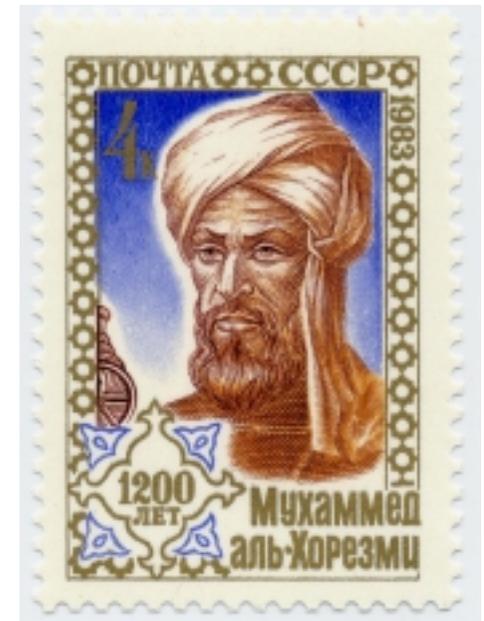
6.5

This algorithm calculates the function

$$f(n) = 8n + 3 \quad \text{for each integer } n.$$

$$\boxed{\text{int } n} \rightarrow \boxed{n := 8 \star n} \rightarrow \boxed{n := n + 3} \rightarrow \boxed{\text{return } n}$$
$$\quad\quad\quad\quad 1 \quad\quad\quad\quad 2$$
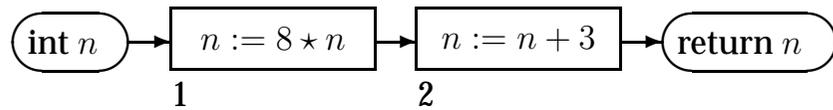
6.6

This algorithm calculates the function

$$f(x) = x^2 \quad \text{for each real } x.$$

$$\boxed{\text{real } x} \rightarrow \boxed{x := x \star x} \rightarrow \boxed{\text{return } x}$$
$$\quad\quad\quad\quad 1$$
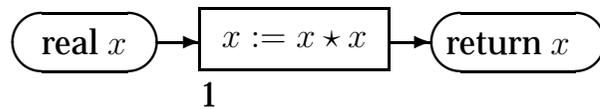
6.7

To calculate x^3, we need to multiply x by x,
then again by x.
This needs two multiplications, and we need to store the
value of x for the second multiplication.
We do this by introducing a new variable u to store the
value.

$$\boxed{\text{real } x} \rightarrow \boxed{u := x} \rightarrow \boxed{x := x \star x} \rightarrow \boxed{x := x \star u} \rightarrow \boxed{\text{return } x}$$
$$\quad\quad\quad\quad 1 \quad\quad\quad\quad 2 \quad\quad\quad\quad 3$$
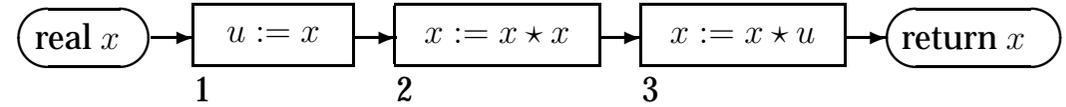
6.8

To calculate $x^2 + x^3$, we need to calculate x^2 and x^3,
and then add them:

$$\boxed{\text{real } x} \rightarrow \boxed{u := x} \rightarrow \boxed{x := x \star x} \rightarrow \boxed{u := x \star u}$$
$$\quad\quad\quad\quad 1 \quad\quad\quad\quad 2 \quad\quad\quad\quad 3$$
$$\boxed{x := x + u} \rightarrow \boxed{\text{return } x}$$
$$\quad\quad 4$$
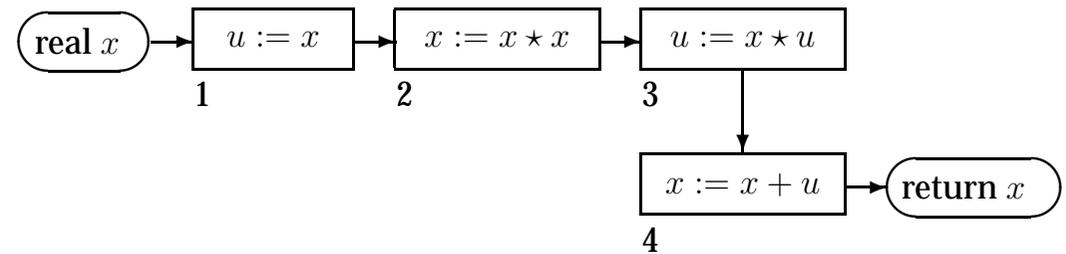
6.9

To check that this algorithm makes a correct calculation, we write x_i for the number assigned to x after the i-th box has been performed.

So from the diagram we can read off the equations

$$x_1 = x; \qquad u_1 = x;$$
$$x_2 = x_1 \star x_1; \quad u_2 = u_1;$$
$$x_3 = x_2; \qquad u_3 = x_2 \star u_2;$$
$$x_4 = x_3 + u_3; \quad u_4 = u_3;$$
$$\text{return } x_4.$$

6.10

Solving these equations tells us what number is returned.

$$
\begin{aligned}
x_4 &= x_3 + u_3 \\
&= x_2 + (x_2 \star u_2) \\
&= (x_1 \star x_1) + ((x_1 \star x_1) \star u_1) \\
&= (x \star x) + ((x \star x) \star x) \\
&= x^2 + x^3
\end{aligned}
$$

So the algorithm is correct.

6.11

To calculate the function of two variables

$$f(x, y) = xy,$$

we need two inputs x and y, and we multiply them:
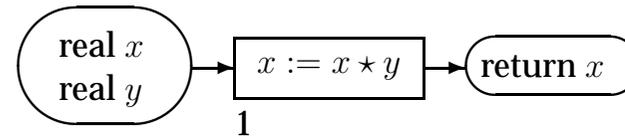
6.12

To calculate the absolute value $|x|$, we have two cases according to whether x is $\geqslant 0$ or not. This needs a test.
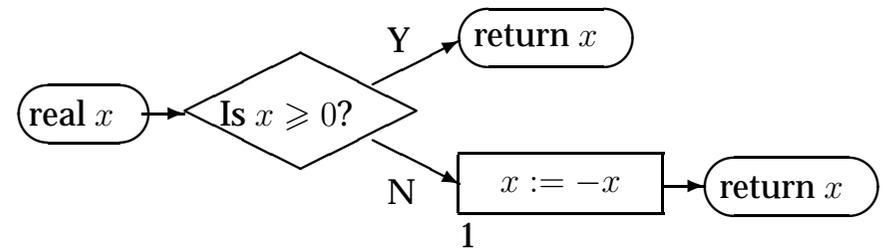
Checking the correctness of this algorithm is more complicated because of the diamond. We won't go into the details.

6.13

We write $n!$ (pronounced 'n factorial') for

$$n(n-1)(n-2)\ldots 2.1,$$

i.e. the result of multiplying together all the integers from 1 to n.

The clue to calculating $n!$ is its *recursive definition*, which defines $(n+1)!$ in terms of $n!$:

$$1! = 1,$$
$$(n+1)! = (n+1) \times n! \quad \text{for each } n \geqslant 1.$$

14

6.14

Consider the formula

$$(n+1)! = (n+1) \times n!.$$

Dividing both sides by $n!$ gives

$$\frac{(n+1)!}{n!} = \frac{(n+1) \times n!}{n!} = n+1.$$

15

6.15

For example, taking $n = 12$, we have $n+1 = 13$ and so

$$\frac{13!}{12!} = 13.$$

Or turning the fraction upside down,

$$\frac{12!}{13!} = \frac{1}{13}.$$

16

6.16

Since the formula holds for any number $n \geqslant 1$, we can use a different letter instead:

$$(m+1)! = (m+1) \times m!.$$

Putting $m = n-1$, we have $m+1 = n-1+1 = n$, so

$$n! = n \times (n-1)!.$$

17

So we can substitute this value for $n!$ in the previous formula:

$$
\begin{aligned}
(n+1)! &= (n+1) \times n! \\
&= (n+1) \times (n \times (n-1)!) \\
&= (n+1) \times n \times (n-1)!.
\end{aligned}
$$

So dividing by $(n-1)!$,

$$
\frac{(n+1)!}{(n-1)!} = (n+1) \times n.
$$

The same argument over again gives

$$
\frac{(n+1)!}{(n-2)!} = (n+1) \times n \times (n-1).
$$

Putting in some sample numbers, we get

$$
\frac{50!}{48!} = 50 \times 49; \quad \frac{50!}{47!} = 50 \times 49 \times 48.
$$

Some sample values:

$$
1! = 1, \ 2! = 2, \ 3! = 6, \ 4! = 24, \ 5! = 120, \ 6! = 720, \ 7! = 5040.
$$

When $n = 0$, the formula $(n+1)! = (n+1) \times n!$ **gives**

$$
1! = 1 \times 0!,
$$

so we put $0! = 1$.

For example

$$
\begin{aligned}
5! &= (4+1)! = 5 \times 4! \\
&= 5 \times (3+1)! = 5 \times (4 \times 3!) \\
&= 5 \times (4 \times (2+1)!) = 5 \times (4 \times (3 \times 2!)) \\
&= 5 \times (4 \times (3 \times (1+1)!)) = 5 \times (4 \times (3 \times (2 \times 1!))) \\
&= 5 \times (4 \times (3 \times (2 \times 1))).
\end{aligned}
$$

To calculate this, the algorithm will go around a loop. The number of times round the loop is controlled by the numbers in the calculation.

The function $f(n) = n!$, where n is a positive integer.
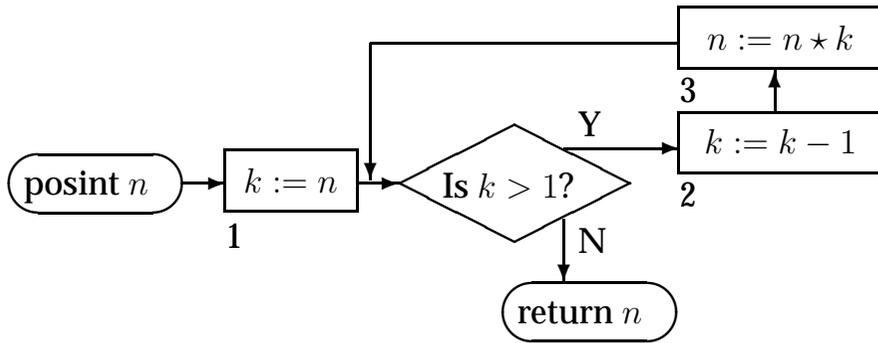
To check this calculation, we write $x_{i,j}$ for the value at x after box i has been performed for the j-th time. Thus:

$n_1 = n;$ $\qquad k_1 = n;$

$$n_{2,j} = \begin{cases} n_1 & \text{if } j = 1, \\ n_{3,j-1} & \text{if } j > 1 \end{cases} \;;\quad k_{2,j} = \begin{cases} k_1 - 1 & \text{if } j = 1, \\ k_{3,j-1} - 1 & \text{if } j > 1 \end{cases}$$

$n_{3,j} = n_{2,j} \times k_{2,j};$ $\qquad k_{3,j} = k_{2,j};$

return $n_{3,j}$ when $k_{3,j} \leqslant 1$.

To check these equations needs special methods.
Instead we shall walk through the calculation when $n = 3$:

$$n = 3;$$
$$n_1 = 3; \qquad k_1 = 3;$$
$$n_{2,1} = 3; \quad k_{2,1} = 2;$$
$$n_{3,1} = 6; \quad k_{3,1} = 2;$$
$$n_{2,2} = 6; \quad k_{2,2} = 1;$$
$$n_{3,2} = 6; \quad k_{3,2} = 1;$$
$$\text{return } 6.$$

Polynomial functions

Polynomial functions are the functions from reals to reals that we can calculate using algorithms as above, but no loops.

To express a polynomial function, we need to use some variables x, y etc. for the inputs.

For example the following are polynomial functions:

- The functions $f(x) = x$, $f(x, y) = x$, $f(x, y) = y$ etc.

- Constant functions such as $f(x, y, z) = 13.7$.

6.25

If $f(x, y)$ and $g(x, y)$ are polynomial functions, then so are the functions

$$\begin{aligned} h(x, y) &= f(x, y) + g(x, y), \\ j(x, y) &= f(x, y) - g(x, y), \\ k(x, y) &= f(x, y).g(x, y). \end{aligned}$$

And likewise with any other variables.

So we can add, subtract and multiply polynomial functions.

6.26

Polynomials are expressions used to express polynomial functions.

A *polynomial in the variable* x is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where $a_n, \ldots, a_0$ are real numbers and $a_n \neq 0$.
We call a_i the *coefficient* of x^i.
The expressions $a_i x^i$ are the *terms* of the polynomial.
We call n the *degree* of the polynomial.

6.27

We also count

$$0$$

as a polynomial of degree $-\infty$.

Polynomials of degree $\leqslant 1$ are called *linear* polynomials, polynomials of degree $\leqslant 2$ are said to be *quadratic*, and polynomials of degree $\leqslant 3$ are called *cubic*.

6.28

Examples

Polynomials of degree 0:

$$5, \; 43, \; -1, \; \pi$$

Polynomials of degree 1:

$$3x + 2, \; 7x + 0, \; -4x + 1$$

We write $7x + 0$ as $7x$ for short.
Polynomials of degree 2:

$$1x^2 - 3x + 5, \; 2x^2 - 2.$$

6.29

To see that every polynomial function of one variable can be expressed by a polynomial in x,
we need to check five things:

(a) x is a polynomial in x. (Clear.)

(b) Each real number is a polynomial in x. (Clear; use polynomials of degree 0 and the polynomial 0.)

6.30

(c) The sum of two polynomials in x is a polynomial in x.

(d) Subtracting one polynomial in x from another gives a polynomial in x.

(e) Multiplying two polynomials in x gives a polynomial in x.

We will check (c)–(e) by seeing how to do these calculations.

6.31

To add polynomials, we add the coefficients of each power of x:

$$
\begin{array}{rrrrr}
 & 2x^3 & -5x^2 & +x & -3 \\
+ & & 6x^2 & & -4 \\
\hline
= & 2x^3 & +x^2 & +x & -7
\end{array}
$$

Subtracting is similar;
we subtract separately at each power of x.

6.32

Multiplying is harder. We multiply each term of the first polynomial by each term of the second, and then we add together the coefficients for each power of x.

$$
\begin{aligned}
(3x^2 + 4x - 1)(5x^3 + 2x^2) &= (15x^5 + 20x^4 - 5x^3) \\
&\quad + (6x^4 + 8x^3 - 2x^2) \\
&= 15x^5 + 26x^4 + 3x^3 + 8x^3 - 2x^2.
\end{aligned}
$$

6.33

How do we calculate $(x + y)^4$?

$$(x + y)$$
$$(x + y)$$
$$(x + y)$$
$$(x + y)$$
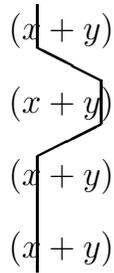
The line picks out x or y from each of the four factors.
It picks out x 3 times and y 1 time, so it gives a term x^3y.
How many other ways of getting x^3y are there in this product?

6.34

The number of ways of getting x^3y in $(x + y)^4$ is
the number of ways of choosing the three factors
that contribute x,
i.e. the number of ways of choosing a subset of 3 things from
a set of 4 things.

This number is called 'four choose three', in symbols

$$C(4, 3) \quad \text{or} \quad \binom{4}{3}$$

6.35

The general formula for $(x + y)^n$, where n is a positive integer, is

$$
\begin{aligned}
(x + y)^n \;=\; & C(n, n)x^n + C(n, n - 1)x^{n-1}y + \ldots \\
& + C(n, 1)xy^{n-1} + C(n, 0)y^n.
\end{aligned}
$$

This is called the *binomial formula*, and the numbers $C(i, j)$ are called *binomial coefficients*.

6.36

Pascal's triangle is the array of numbers

$$
\begin{array}{ccccccccc}
& & & & 1 & & & & \\
& & & 1 & & 1 & & & \\
& & 1 & & 2 & & 1 & & \\
& 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
\end{array}
$$

$$
\begin{array}{ccccccccccc}
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

6.37

The numbers in Pascal's triangle are the binomial coefficients:

$$
\begin{array}{ccccccccc}
 & & & & C(0,0) & & & & \\
 & & & C(1,1) & & C(1,0) & & & \\
 & & C(2,2) & & C(2,1) & & C(2,0) & & \\
 & C(3,3) & & C(3,2) & & C(3,1) & & C(3,0) & \\
C(4,4) & & C(4,3) & & C(4,2) & & C(4,1) & & C(4,0) \\
 & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}
$$

6.38

Blaise Pascal

1623–1687

6.39

We can form polynomials in several variables too, to express polynomial functions with several inputs.

For example

$$7s^2t^3 + 5s^4 - 6st - 66$$

is a polynomial in the variables s, t.
The term $7s^2t^3$ has degree $2 + 3 = 5$.
Since no term has higher degree,
the polynomial has degree 5.