## MTH5118 Probability II Test Solutions 2009

1. $\quad X$ and $Y$ are independent random variables with $X \sim \operatorname{Binomial}\left(2, \frac{1}{3}\right)$ and $Y \sim \operatorname{Bernoulli}\left(\frac{1}{3}\right)$.
(a) (6 points) Write down the probability generating functions $G_{X}(t)$ and $G_{Y}(t)$.
$G_{X}(t)=\left(\frac{1}{3} t+\frac{2}{3}\right)^{2}$ and $G_{Y}(t)=\frac{1}{3} t+\frac{2}{3}$
(b) (4 points) Obtain the probability generating function of $Z=X+Y$.
$G_{Z}(t)=G_{X}(t) G_{Y}(t)=\left(\frac{1}{3} t+\frac{2}{3}\right)^{2}\left(\frac{1}{3} t+\frac{2}{3}\right)=\left(\frac{1}{3} t+\frac{2}{3}\right)^{3}$
(c) (8 points) Obtain the probability mass function for $Z$.

The p.g.f. for a $\operatorname{Binomial}(n, p)$ r.v. is given by $(q+p t)^{n}$. Hence $Z \sim \operatorname{Binomial}\left(3, \frac{1}{3}\right)$ and so

$$
P(Z=k)=\binom{3}{k}\left(\frac{1}{3}\right)^{k}\left(\frac{2}{3}\right)^{3-k}, \quad 0 \leq k \leq 3
$$

(and $P(Z=z)=0$ for all other values of $z$ ).
2. Let $X$ be a random variable with probability generating function

$$
G_{X}(t)=\frac{e^{t-1}}{3-2 t}
$$

(a) (12 points) Differentiate the p.g.f. to obtain $E[X]$ and $\operatorname{Var}(X)$.
$G_{X}^{\prime}(t)=\frac{e^{t-1}}{3-2 t}+2 \frac{e^{t-1}}{(3-2 t)^{2}}$
$G_{X}^{\prime \prime}(t)=\frac{e^{t-1}}{3-2 t}+4 \frac{e^{t-1}}{(3-2 t)^{2}}+8 \frac{e^{t-1}}{(3-2 t)^{3}}$
Therefore $E[X]=G_{X}^{\prime}(1)=3$ and $E[X(X-1)]=G_{X}^{\prime \prime}(1)=13$ and so
$\operatorname{Var}(X)=E[X(X-1)]+E[X]-(E[X])^{2}=13+3-9=7$.
(b) (6 points) Find $P(X=0), P(X=1)$ and $P(X=2)$.

We use the formula $P(X=k)=\frac{G_{X}^{(k)}(0)}{k!}$. Thus

$$
\begin{aligned}
& P(X=0)=G_{X}(0)=\frac{e^{-1}}{3}, \\
& P(X=1)=G_{X}^{\prime}(0)=e^{-1}\left(\frac{1}{3}+\frac{2}{9}\right)=\frac{5}{9} e^{-1}, \\
& P(X=2)=\frac{G_{X}^{\prime \prime}(0)}{2}=\frac{e^{-1}}{2}\left(\frac{1}{3}+\frac{4}{9}+\frac{8}{27}\right)=\frac{29}{54} e^{-1}
\end{aligned}
$$

3. (12 points) Paul enters the Crazy Casino with 20 pounds and on each spin of a roulette wheel he bets 2 pounds at even odds. At the Crazy Casino the probability of winning is $2 / 3$ and the probability of losing is $1 / 3$. If he ever possesses only 12 pounds he will leave, and if he ever possesses 24 pounds he also chooses to leave. What is the probability that Paul leaves with 24 pounds?
In our case 1 unit is $£ 2$, and therefore $k=\frac{20}{2}=10, N=\frac{24}{2}=12, M=\frac{12}{2}=6$ and we want to find $r_{10}$. Since $p \neq q, r_{k}=\frac{\left(\frac{q}{p}\right)^{k}-\left(\frac{q}{p}\right)^{M}}{\left(\frac{q}{p}\right)^{N}-\left(\frac{q}{p}\right)^{M}}$.
Therefore the probability that Paul leaves with $£ 24$ is $r_{10}=\frac{\left(\frac{1}{2}\right)^{10}-\left(\frac{1}{2}\right)^{6}}{\left(\frac{1}{2}\right)^{12}-\left(\frac{1}{2}\right)^{6}}=\frac{\left(\frac{1}{2}\right)^{4}-1}{\left(\frac{1}{2}\right)^{6}-1}=\frac{20}{21}$.
4. (15 points) (a) State the total probability formula for expectations.
(b) John plays a series of independent games. In each game he pays $£ 1$ and tosses a fair six sided die. If $k=1$ is obtained then he receives $£ C$. If $k=2,3$ or 4 then he does not receive anything and the game continues. The game stops if 5 or 6 is obtained and also in the last game no money is paid to John. Let $X$ be the amount of his gain by the end of the series. Find $E[X]$.
(c) The game is said to be fair if the average value of the gain is zero: $E[X]=0$. What value of $C$ would make this game fair?
SOLUTION
(a) If $B_{1}, B_{2}, B_{3}, \ldots$ is a partition of the sample space $\Omega$ and $X$ is a random variable defined on $S$ then $E[X]=\sum_{j} E\left[X \mid B_{j}\right] P\left(B_{j}\right)$
(b) Let $B_{1}$ be the events that at the first game $k=1, B_{2}$ that at the first game $k=2,3$ or 4 , and $B_{3}$ that at the first game $k=5$ or 6 . This is a partition. Then $P\left(B_{1}\right)=\frac{1}{6}, P\left(B_{2}\right)=\frac{1}{2}$, $P\left(B_{3}\right)=\frac{1}{3}$. Next, $E\left[X \mid B_{1}\right]=C-1+E[X], E\left[X \mid B_{2}\right]=-1+E[X], E\left[X \mid B_{3}\right]=-1$. Hence

$$
\begin{aligned}
E[X] & =\sum_{j=1}^{3} E\left[X \mid B_{j}\right] P\left(B_{j}\right)=(C-1+E[X]) \frac{1}{6}+(-1+E[X]) \frac{1}{2}+(-1) \frac{1}{3} \\
& =\frac{C}{6}-1+\frac{2}{3} E[X]
\end{aligned}
$$

Therefore $\frac{1}{3} E[X]=\frac{C}{6}-1$ and so $E[X]=\frac{C}{2}-3$.
(c) $E[X]=0=\frac{C}{2}-3$. Hence $C=6$.
5. In a particular society each male has no children at all with probability $\frac{1}{4}$, has one son with probability $\frac{3}{8}$ and two sons with probability $\frac{3}{8}$.
(a) (10 points) Find the probability that the male line of descent of a particular individual dies out by generation 2 .
The p.g.f. of the generating r.v. in this case is $G(t)=\frac{1}{4}+\frac{3}{8} t+\frac{3}{8} t^{2}$. Obviously, $\theta_{1}=\frac{1}{4}$. We know that $\theta_{n+1}=G\left(\theta_{n}\right)$. Hence $\theta_{2}=G\left(\theta_{1}\right)=\frac{1}{4}+\frac{3}{8} \frac{1}{4}+\frac{3}{8}\left(\frac{1}{4}\right)^{2}=\frac{47}{128}$.
(b) (10 points) Find the probability that the male line of descent of a particular individual will die out eventually.
$\theta$ is the smallest positive root of $G(t)=t$. So we solve $\frac{1}{4}+\frac{3}{8} t+\frac{3}{8} t^{2}=t$, or equivalently $3 t^{2}-5 t+2=0$. This is simple to solve as we know that $t=1$ is a root. We have $(t-1)(3 t-2)=0$ giving solutions $t=1$ and $t=\frac{2}{3}$ and so $\theta=\frac{2}{3}$.
(c) (5 points) State the probability of eventual extinction if initially there are $k$ ancestors forming generation zero.
The probability of eventual extinction in this case is $\theta^{k}=\left(\frac{2}{3}\right)^{k}$
6. (12 points) The number of customers arriving at a shop during one day is a r.v. $N \sim$ Poisson(100). The customers act independently of each other spending a random amount of $X_{j}$ pounds in the shop. Given that $E\left(X_{j}\right)=5, \operatorname{Var}\left(X_{j}\right)=10$, find the average value and the variance of the daily cash flow $S$ in this shop.
We deal with a sum of a random number of random variables $S=X_{1}+X_{2}+\ldots+X_{N}$, where $E(N)=100, \operatorname{Var}(N)=100$. Thereore we have:

$$
\begin{gathered}
E(S)=E(X) \times E(N)=5 \times 100=500 \\
\operatorname{Var}(S)=\operatorname{Var}(X) \times E(N)+[E(X)]^{2} \times \operatorname{Var}(N)=10 \times 100+25 \times 100=3500
\end{gathered}
$$

