Probability II. Solutions to Problem Sheet 9.

Part 1.

1. Since X and Y are independent,

$$f_{X,Y}(x,y) = \theta^2 x e^{-\theta x} \theta^2 y e^{-\theta y} = \theta^4 xy e^{-\theta(x+y)}$$

for $x > 0$ and $y > 0$. Since $U = X$ and $V = X + Y$, the inverses are $X = U$ and $Y = V - U$. Then

$$f_{U,V}(u,v) = \theta^4 u(v-u)e^{-\theta v} \times |\det \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}| = \theta^4 u(v-u)e^{-\theta v}$$

The ranges $x > 0$ and $y > 0$ become $u > 0$ and $v - u > 0$, i.e. $0 < u < v < \infty$. Hence

$$\begin{aligned} f_V(v) &= \int_0^v \theta^4 u(v-u)e^{-\theta v} du = \left[\theta^4 e^{-\theta v}\left(\frac{vu^2}{2} - \frac{u^3}{3}\right)\right]_{u=0}^{u=v} = \theta^4 e^{-\theta v}\left(\frac{v^3}{2} - \frac{v^3}{3}\right) \\ &= \frac{\theta^4 v^3 e^{-\theta v}}{3!} \end{aligned}$$

Therefore $V \sim Gamma(\theta, 4)$. Then (for each $v > 0$),

$$f_{U|V}(u|v) = \frac{f_{U,V}(u,v)}{f_V(v)} = \frac{6u(v-u)}{v^3}$$

for $0 < u < v$.

2. Since $X \sim Exp(1)$ independent of $Y \sim Exp(1)$, $f_{X,Y}(x,y) = e^{-(x+y)}$ for $x > 0$ and $y > 0$.

Since $U = \frac{X}{Y}$ and $V = X + Y$, $X = UY$ so that $V = Y(1+U)$. Therefore $Y = \frac{V}{(1+U)}$ and $X = \frac{UV}{(1+U)} = V\left(1 - \frac{1}{(1+U)}\right)$. Then the ranges $x > 0$ and $y > 0$ become $\frac{uv}{(1+u)} > 0$ and $\frac{v}{(1+u)} > 0$, so that $u > 0$ (hence $(1+u) > 0$) and therefore $v > 0$. Now

$$\det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det \begin{pmatrix} \frac{v}{(1+u)^2} & \frac{u}{(1+u)} \\ -\frac{v}{(1+u)^2} & \frac{1}{(1+u)} \end{pmatrix} = \frac{v}{(1+u)^2}$$

Therefore, for $u > 0$ and $v > 0$,

$$f_{U,V}(u,v) = e^{-v} \times \frac{v}{(1+u)^2} = (ve^{-v}) \times ((1+u)^{-2})$$

So the ranges are not dependent and the joint p.d.f. splits as shown into $g(u) = (1+u)^{-2}$ times $h(v) = ve^{-v}$. Therefore U and V are independent.

Then $f_U(u) = C(1+u)^{-2}$ for $u > 0$ and $f_V(v) = \frac{1}{C}ve^{-v}$ for $v > 0$. The latter is clearly the p.d.f. of a $Gamma(1,2)$ distribution so that $C = 1$. Hence $f_U(u) = (1+u)^{-2}$ for $u > 0$ and $f_V(v) = ve^{-v}$ for $v > 0$.

Part 2.

3. $X|Y = y \sim Exp(y)$. Hence $E[X|Y = y] = \frac{1}{y}$ and $Var(X|Y = y) = \frac{1}{y^2}$. Now $E[X] = E[E[X|Y]] = E[Y^{-1}]$ and

$$\begin{aligned} Var(X) &= E[Var(X|Y)] + Var(E[X|Y]) \\ &= E[Y^{-2}] + Var(Y^{-1}) = E[Y^{-2}] + E[Y^{-2}] - (E[Y^{-1}])^2 \\ &= 2E[Y^{-2}] - (E[Y^{-1}])^2 \end{aligned}$$

So we need to find $E[Y^{-1}]$ and $E[Y^{-2}]$. Using the result that a gamma p.d.f. integrates to one, we obtain

$$E[Y^{-1}] = \int_0^\infty y^{-1}\frac{\theta^3 y^2 e^{-\theta y}}{2}dy = \frac{\theta}{2}\int_0^\infty \theta^2 y e^{-\theta y}dy = \frac{\theta}{2}$$

$$E[Y^{-2}] = \int_0^\infty y^{-2}\frac{\theta^3 y^2 e^{-\theta y}}{2}dy = \frac{\theta^2}{2}\int_0^\infty \theta e^{-\theta y}dy = \frac{\theta^2}{2}$$

Therefore $E[X] = \frac{\theta}{2}$ and $Var(X) = 2\left(\frac{\theta^2}{2}\right) - \left(\frac{\theta}{2}\right)^2 = \frac{3\theta^2}{4}$

The joint p.d.f. for X and Y is just

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y) = ye^{-yx}\frac{\theta^3 y^2 e^{-\theta y}}{2}$$

for $x > 0$ and $y > 0$. Hence, for $x > 0$, using the result that a gamma p.d.f. integrates to one,

$$f_X(x) = \int_0^\infty \frac{\theta^3 y^3 e^{-(\theta+x)y}}{2} dy = \frac{3\theta^3}{(\theta+x)^4} \int_0^\infty \frac{(\theta+x)^4 y^3 e^{-(\theta+x)y}}{3!} dy = \frac{3\theta^3}{(\theta+x)^4}$$

Hence $f_X(x) = \frac{3\theta^3}{(\theta+x)^4}$ for $x > 0$. Now

$$E[X+\theta] = \int_0^\infty \frac{3\theta^3}{(\theta+x)^3} dx = \left[-\frac{3\theta^3}{2(\theta+x)^2} \right]_{x=0}^{x=\infty} = \frac{3\theta}{2}$$

$$E[(X+\theta)^2] = \int_0^\infty \frac{3\theta^3}{(\theta+x)^2} dx = \left[-\frac{3\theta^3}{(\theta+x)} \right]_{x=0}^{x=\infty} = 3\theta^2$$

So $E[X] = E[X+\theta] - \theta = \frac{\theta}{2}$ and

$$Var(X) = Var(X+\theta) = E[(X+\theta)^2] - (E[X+\theta])^2 = 3\theta^2 - \frac{9\theta^2}{4} = \frac{3\theta^2}{4}$$

These are the same results as obtained previously.

4. $M_X(s) = M_{X,Y}(s,0) = \frac{1}{(1-s)^2}$. You can either recognise this as the m.g.f. of $Gamma(1,2)$ and state the mean and variance ($E[X] = 2$ and $Var(X) = 2$) or differentiate the m.g.f. to obtain the results. $M'_X(s) = 2(1-s)^{-3}$ so that $E[X] = M'_X(0) = 2$ and $M''_X(s) = 6(1-s)^{-4}$ so that $E[X^2] = M''_X(0) = 6$ and hence $Var(X) = 6 - 4 = 2$.

$M_Y(t) = M_{X,Y}(0,t) = \frac{1}{(1-t^2)}$. This is the m.g.f. for a double exponential but we haven't found the mean and variance previously. $M'_Y(t) = 2t(1-t^2)^{-2}$ so that $E[Y] = M'_Y(0) = 0$ and $M''_Y(t) = 2(1-t^2)^{-2} + 8t^2(1-t^2)^{-3}$ so that $E[Y^2] = M''_Y(0) = 2$ and hence $Var(X) = 2 - 0 = 2$.

$$\frac{\partial^2 M_{X,Y}(s,t)}{\partial s \partial t} = \frac{\partial}{\partial s} 2t((1-s)^2 - t^2)^{-2} = 8t(1-s)((1-s)^2 - t^2)^{-3}$$

Hence $E[XY] = 0$. Therefore $Cov(X,Y) = E[XY] - E[X]E[Y] = 0$ and so $\rho(X,Y) = 0$. Observe that in this case X and Y are not independent but have coefficient of correlation zero.

$$\begin{aligned}
M_{U,V}(s,t) &= E[e^{s(X+Y)+t(X-Y)}] = E[e^{(s+t)X+(s-t)Y}] \\
&= M_{X,Y}(s+t, s-t) = \frac{1}{(1-s-t)^2-(s-t)^2} \\
&= \frac{1}{(1-s-t+s-t)(1-s-t-(s-t))} = \frac{1}{(1-2t)(1-2s)} \\
&= ((1-2s)^{-1})((1-2t)^{-1})
\end{aligned}$$

So the joint m.g.f. splits into (a function of s only) times (a function of t only). Therefore U and V are independent and $M_U(s) = M_{U,V}(s,0) = (1-2s)^{-1}$ so that $U \sim Exp(1/2)$ and $M_V(t) = M_{U,V}(0,t) = (1-2t)^{-1}$ so that $V \sim Exp(1/2)$.