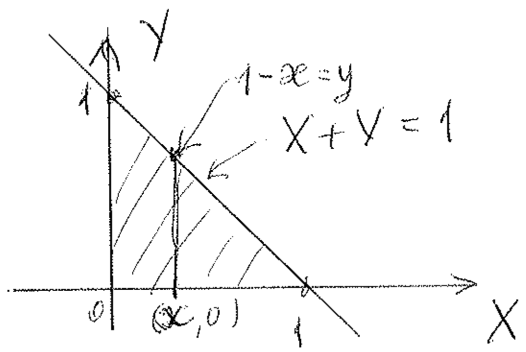
Probability II. Solutions to Problem Sheet 8.

Part 1

1. We integrate over y first, for fixed x, and then over x.



$$
\begin{aligned}
P(X+Y\le 1) &= \int_0^1\left(\int_0^{1-x}\frac{12}{7}(x^2+xy)dy\right)dx = \int_0^1\left[\frac{12}{7}(x^2y+\frac{1}{2}xy^2)\right]_{y=0}^{y=1-x}dx \\
&= \int_0^1\frac{12}{7}\left(x^2(1-x)+\frac{1}{2}x(1-x)^2\right)dx = \int_0^1\frac{6}{7}(x-x^3)dx = \frac{3}{14}
\end{aligned}
$$

2.

(a) $f_{X,Y}(x,y) = C(1+x+xy)$ for $0<x<1$ and $0<y<1$ so ranges are not dependent but the joint p.d.f. cannot be written as a function of x times a function of y. Hence X and Y are not independent.

(b) $f_{X,Y}(x,y) = Ce^{-x-3y} = (e^{-x})(Ce^{-3y})$ for $0<x<\infty$ and $0<y<\infty$. The ranges are not dependent and the joint p.d.f. can be written as $g(x)h(y)$ where $g(x)=e^{-x}$ and $h(y)=Ce^{-3y}$. So X and Y are independent.

Now for some constant K, $f_X(x) = Kg(x) = Ke^{-x}$ for $0<x<\infty$ and $f_Y(y) = \frac{1}{K}h(y) = \frac{C}{K}e^{-3y}$ for $0<y<\infty$, where K and C are such that each of the marginal p.d.f.'s integrate to 1.

It is easily seen that $X\sim Exp(1)$ so that $K=1$ and $Y\sim Exp(3)$ so that $\frac{C}{K}=3$ and hence $C=3$.

(c) $f_{X,Y}(x,y) = Cx^2(1+y)$ for $x>0$, $y>0$ and $x+y<1$. The ranges are dependent so that X and Y are not independent.

(d) $f_{X,Y}(x,y) = C\frac{xe^{-2x}}{y^2}$ for $0<x<\infty$ and $1<y<\infty$. The ranges are not dependent and the joint p.d.f. can be written as $g(x)h(y)$ where $g(x)=xe^{-2x}$ and $h(y)=\frac{C}{y^2}$. So X and Y are independent.

Now for some constant K, $f_X(x) = Kg(x) = Kxe^{-2x}$ for $0 < x < \infty$ and $f_Y(y) = \frac{1}{K}h(Y) = \frac{C}{Ky^2}$ for $1 < y < \infty$, where K and C are such that each of the marginal p.d.f.'s integrate to 1.

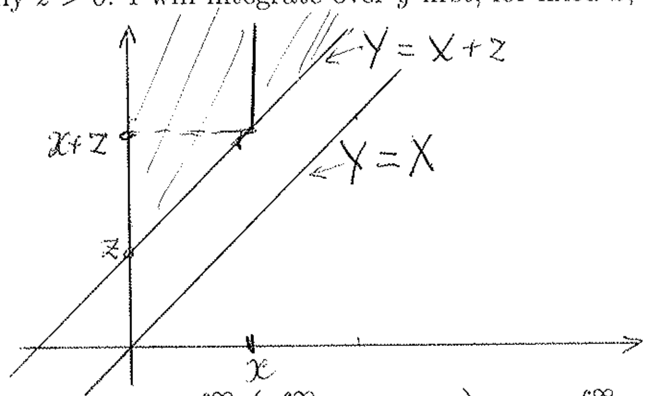
It is easily seen that $X \sim Gamma(1, 2)$ so that $K = \frac{2^2}{1!} = 4$. Also

$$1 = \int_1^\infty \frac{C}{K} y^{-2} dy = \left[-\frac{C}{K} y^{-1}\right]_{y=1}^{\infty} = \frac{C}{K}$$

Hence $C = K = 4$ and $f_Y(y) = y^{-2}$ for $1 < y < \infty$ and the p.d.f. is zero elsewhere.

Part 2

3. Consider any $z > 0$. I will integrate over y first, for fixed x, and then over x.



$$\begin{aligned} P(Y - X > z) &= \int_0^\infty \left(\int_{x+z}^\infty 2e^{-(x+y)} dy \right) dx = \int_0^\infty \left[-2e^{-(x+y)}\right]_{y=x+z}^{y=\infty} dx \\ &= \int_0^\infty 2e^{-(2x+z)} dx = \left[-e^{-(2x+z)}\right]_{x=0}^{x=\infty} = e^{-z} \end{aligned}$$

Therefore $F_Z(z) = P(Z \leq z) = 1 - P(Z > z) = 1 - e^{-z}$ for $z > 0$. Also $F_Z(z) = 0$ for $z \leq 0$. Hence $f_Z(z) = \frac{dF_Z(z)}{dz} = e^{-z}$ for $z > 0$ and $f_Z(z) = 0$ elsewhere. So $Z \sim Exp(1)$.

4. $M_X(t) = (1 - 2t)^{-n/2}$ and $M_Y(t) = (1 - 2t)^{-n/2}$.

(a) If $U = X + Y$, then

$$M_U(t) = E[e^{t(X+Y)}] = E[e^{tX}]E[e^{tY}] = M_X(t)M_Y(t) = (1 - 2t)^{-n}$$

This is the m.g.f. of a chi-squared distribution with parameter $2n$ (this can also be stated as $Gamma(\frac{1}{2}, n)$). So U has that distribution.

(b) If $V = X - Y$, then

$$\begin{aligned} M_V(t) &= E[e^{t(X-Y)}] = E[e^{tX}e^{-tY}] = E[e^{tX}]E[e^{-tY}] = M_X(t)M_Y(-t) \\ &= (1-2t)^{-n/2}(1+2t)^{-n/2} = (1-4t^2)^{-n/2} \end{aligned}$$

When $n = 2$, $M_V(t) = (1-4t^2)^{-1}$. For the double exponential with parameter θ, the m.g.f. was $M(t) = \left(1 - \frac{t^2}{\theta^2}\right)^{-1}$. Therefore V has double exponential distribution with parameter $\theta = \frac{1}{2}$.

5. X and Y have joint p.d.f. $f_{X,Y}(x,y) = (x+y)$ for $0 < x < 1$ and $0 < y < 1$. Hence

$$f_Y(y) = \int_0^1 (x+y)dx = \left[\frac{1}{2}x^2 + yx\right]_{x=0}^{x=1} = y + \frac{1}{2}$$

$$E[Y] = \int_0^1 \left(y^2 + \frac{1}{2}y\right) dy = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

$$E[Y^2] = \int_0^1 \left(y^3 + \frac{1}{2}y^2\right) dy = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

So $E[Y] = \frac{7}{12}$ and $Var(Y) = E[Y^2] - (E[Y])^2 = \frac{11}{144}$.

By symmetry of the joint p.d.f. in x and y, X has the same marginal distribution as Y and so $E[X] = \frac{7}{12}$ and $Var(X) = \frac{11}{144}$.

$$E[XY] = \int_0^1 \int_0^1 (x^2y + y^2x)dxdy = \int_0^1 \left(\frac{1}{3}y + \frac{1}{2}y^2\right) dy = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Hence $Cov(X,Y) = E[XY] - E[X]E[Y] = \frac{1}{3} - \frac{49}{144} = \frac{-1}{144}$ and so

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{-\frac{1}{144}}{\frac{11}{144}} = -\frac{1}{11}$$