Probability II. Solutions to Problem Sheet 7.

Part 1. 1. X has p.d.f. $f_X(x) = 2\theta x e^{-\theta x^2}$ for $x > 0$ and $f_X(x)$ is zero elsewhere.

$Y = X^2$ has inverse $X = \sqrt{Y}$. The range for Y has end-points 0 and infinity. Hence for $0 < y < \infty$,

$$f_Y(y) = f_X(\sqrt{y}) \left| \frac{d\sqrt{y}}{dy} \right| = 2\theta\sqrt{y}e^{-\theta y} \times \frac{1}{2\sqrt{y}} = \theta e^{-\theta y}$$

$f_Y(y) = 0$ elsewhere. This is just the p.d.f. of $Exp(\theta)$.

2. $X \sim N(0,1)$ and $Y = |X|$.

$F_Y(y) = 0$ for $y \leq 0$. For $y > 0$

$$F_Y(y) = P(Y \leq y) = P(|X| \leq y) = P(-y \leq X \leq y) = F_X(y) - F_X(-y)$$

Now differentiate with respect to y. For $y > 0$,

$$f_Y(y) = f_X(y) + f_X(-y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2} + \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2} = \frac{2}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}$$

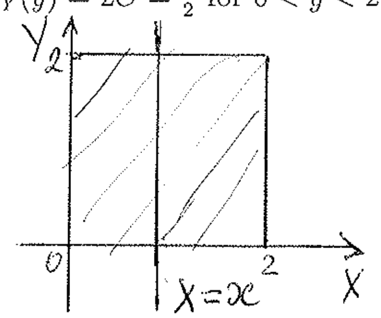
$f_Y(y) = 0$ elsewhere.

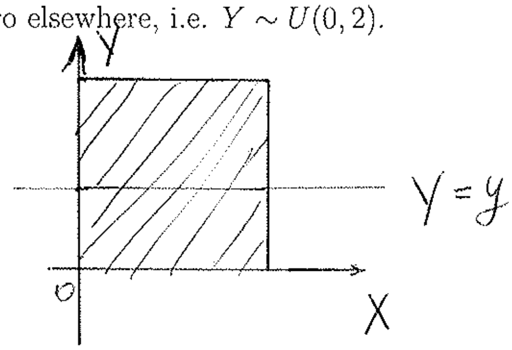
3. Let X and Y have joint p.d.f. $f_{X,Y}(x,y) = C$ for $0 < x < 2$ and $0 < y < 2$

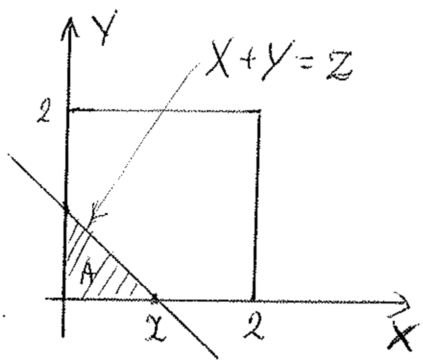
The joint p.d.f. is constant over its support and the area of the support is 4. Hence $1 = 4C$ and so $C = \frac{1}{4}$.

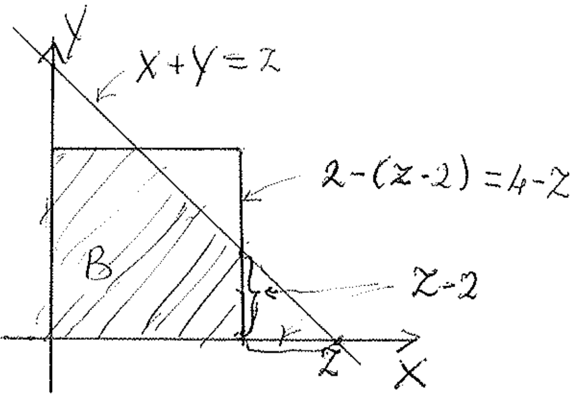
Also $f_X(x)$=area above the line $X = x$ within the support of the joint p.d.f. so that $f_X(x) = 2C = \frac{1}{2}$ for $0 < x < 2$ and is zero elsewhere, i.e. $X \sim U(0,2)$.

Similarly $f_Y(y)$=area above the line $Y = y$ within the support of the joint p.d.f. so that $f_Y(y) = 2C = \frac{1}{2}$ for $0 < y < 2$ and is zero elsewhere, i.e. $Y \sim U(0,2)$.









For (i) $0 < z \leq 2$, $P(X+Y \leq z) = C$ times the area shaded A, so that $P(X+Y \leq z) = C \times \frac{1}{2}z^2 = \frac{1}{8}z^2$.

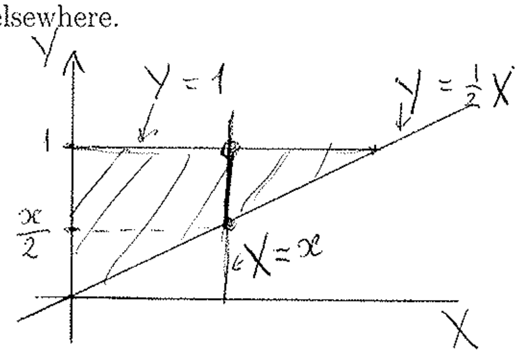
For (ii) $2 < z < 4$, $P(X+Y \leq z) = C$ times the area shaded B, so that $P(X+Y \leq z) = C \times \left(4 - \frac{1}{2}(2-(z-2))^2\right) = 1 - \frac{1}{8}(4-z)^2$.

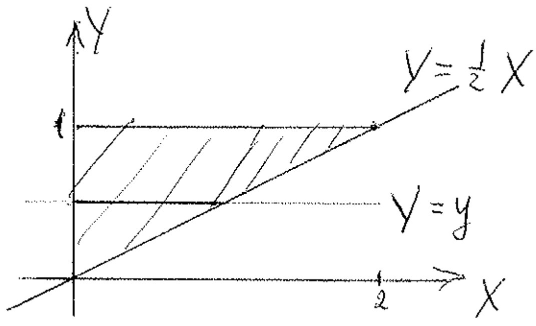
If (iii) $z \leq 0$ then $P(X+Y \leq z) = 0$ and if (iv) $z \geq 4$ then $P(X+Y \leq z) = 1$.

This gives the value of the c.d.f. for Z since $F_Z(z) = P(Z \leq z) = P(X+Y \leq z)$. So differentiating with respect to z we obtain

$f_Z(z) = \frac{1}{4}z$ for $0 < z \leq 2$, $f_Z(z) = \frac{1}{4}(4-z)$ if $2 < z < 4$ and $f_Z(z) = 0$ elsewhere.

4. X and Y have joint p.d.f. $f_{X,Y}(x,y) = C$ for $0 < x < 2y < 2$ and $f_{X,Y}(x,y) = 0$ elsewhere.

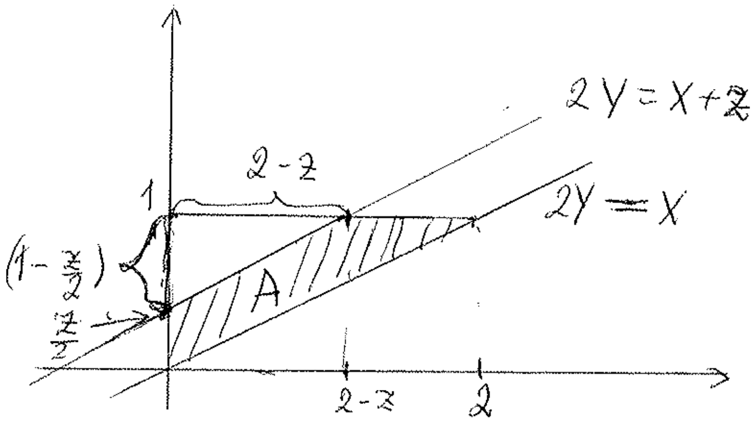




The support of the joint p.d.f. lies in the region between the lines $X = 0$, $2Y = X$ and $2Y = 2$, i.e. $X = 0$, $Y = \frac{X}{2}$ and $Y = 1$. The area of the support is 1. Hence $1 = C \times 1$ so that $C = 1$.

$f_X(x)$=area above the line $X = x$ within the support of the joint p.d.f. For $0 < x < 2$, the length of the line is $\left(1 - \frac{x}{2}\right)$ and hence $f_X(x) = C\left(1 - \frac{x}{2}\right) = \left(1 - \frac{x}{2}\right)$. ($f_X(x) = 0$ elsewhere.)

$f_Y(y)$=area above the line $Y = y$ within the support of the joint p.d.f. For $0 < y < 1$, the length of this line is $2y$ so that $f_Y(y) = C \times 2y = 2y$. ($f_Y(y) = 0$ elsewhere.)



For $0 < z < 2$, $P(2Y - X \le z)$ is just C times the area shaded A. This area is just $1 - \frac{1}{2}\left(1 - \frac{z}{2}\right)(2 - z)$. Hence

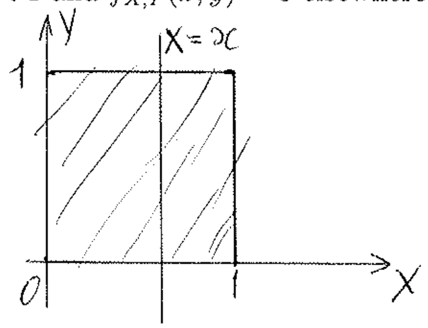
$$P(2Y - X \le z) = C \times \left(1 - \frac{1}{2}\left(1 - \frac{z}{2}\right)(2 - z)\right) = 1 - \frac{1}{2}\left(1 - \frac{z}{2}\right)(2 - z) = 1 - \frac{1}{4}(2 - z)^2$$

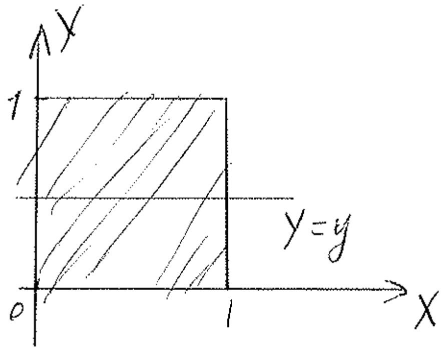
$P(2Y - X \le z) = 0$ if (i) $z \le 0$ and $P(2Y - X \le z) = 1$ if (ii) $z \ge 2$

This gives the value of the c.d.f. for Z since $F_Z(z) = P(Z \le z) = P(2Y - X \le z)$. So differentiating with respect to z we obtain $f_Z(z) = \frac{1}{2}(2 - z) = 1 - \frac{z}{2}$ for $0 < z < 2$ and $f_Z(z) = 0$ elsewhere.

Part 2

5. Random variables X and Y have joint p.d.f. $f_{X,Y}(x, y) = C(x^2 + xy)$ for $0 < x < 1$, $0 < y < 1$ and $f_{X,Y}(x, y) = 0$ elsewhere.





$$f_X(x) = \int_0^1 C(x^2 + xy)dy = \left[C\left(x^2 y + \frac{1}{2}xy^2\right)\right]_{y=0}^{y=1} = C\left(x^2 + \frac{1}{2}x\right)$$

for $0 < x < 1$ and $f_X(x) = 0$ elsewhere.

$$f_Y(y) = \int_0^1 C(x^2 + xy)dx = \left[C\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 y\right)\right]_{x=0}^{x=1} = C\left(\frac{1}{3} + \frac{1}{2}y\right)$$

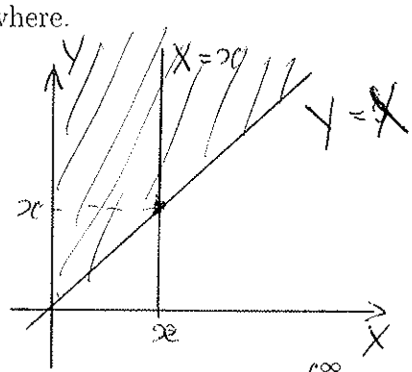
for $0 < y < 1$ and $f_Y(y) = 0$ elsewhere.

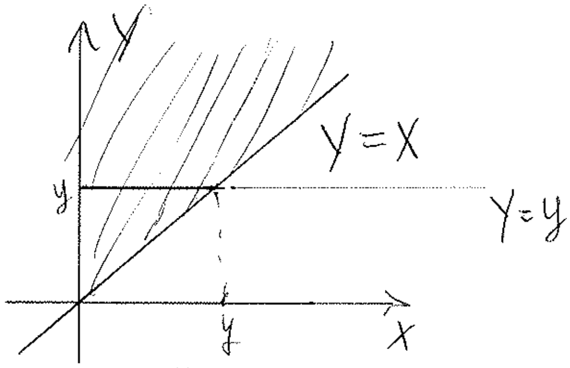
We find C by integrating either marginal p.d.f., e.g.

$$1 = \int_0^1 C\left(\frac{1}{3} + \frac{1}{2}y\right) dy = \left[C\left(\frac{1}{3}y + \frac{1}{4}y^2\right)\right]_{y=0}^{y=1} = \frac{7}{12}C$$

Hence $C = \frac{12}{7}$.

6. X and Y have joint p.d.f. $f_{X,Y}(x,y) = Ce^{-(x+y)}$ for $0 < x < y < \infty$ and $f_{X,Y}(x,y) = 0$ elsewhere.





$$f_X(x) = \int_x^\infty Ce^{-x}e^{-y}dy = Ce^{-x}\left[-e^{-y}\right]_{y=x}^{y=\infty} = Ce^{-2x}$$

for $0 < x < \infty$, and $f_X(x) = 0$ elsewhere.

$$f_Y(y) = \int_0^y Ce^{-x}e^{-y}dx = Ce^{-y}\left[-e^{-x}\right]_{x=0}^{x=y} = Ce^{-y}\left(1 - e^{-y}\right)$$

for $0 < y < \infty$ and $f_Y(y) = 0$ elsewhere.

We find C by integrating either p.d.f. Clearly it is easiest to integrate $f_X(x)$.

$$1 = \int_0^\infty Ce^{-2x}dx = C\left[-\frac{1}{2}e^{-2x}\right]_{x=0}^{x=\infty} = C\frac{1}{2}$$

Hence $C = 2$.