Probability II. Solutions to Problem Sheet 6.

Part 1

1. Find C by using the result that the p.d.f. integrates to one.

$$1 = \int_0^2 C(2x - x^2)dx = \left[C\left(x^2 - \frac{x^3}{3}\right)\right]_{x=0}^{x=2} = C\left(4 - \frac{8}{3}\right) = \frac{4}{3}C$$

Hence $C = \frac{3}{4}$.

$F_X(x) = 0$ for $x \leq 0$, $F_X(x) = 1$ for $x \geq 2$ and for $0 < x < 2$,

$$F_X(x) = \int_0^x \frac{3}{4}(2s - s^2)ds = \frac{3}{4}\left(x^2 - \frac{x^3}{3}\right) = \frac{x^2(3-x)}{4}$$

$$E[X] = \int_0^2 \frac{3}{4}(2x^2 - x^3)dx = \left[\frac{3}{4}\left(\frac{2}{3}x^3 - \frac{x^4}{4}\right)\right]_{x=0}^{x=2} = 4 - 3 = 1$$

If the mean is μ, to show that the median is equal to the mean we just need to show that $F_X(\mu) = \frac{1}{2}$. Here $\mu = 1$ and $F_X(1) = \frac{1^2(3-1)}{4} = \frac{1}{2}$. Hence the mean and median are both equal to one.

2. $M_X(t) = e^{\alpha t}\left(1 - \frac{t}{\theta}\right)^{-1}$. Hence, differentiating by parts,

$$M_X'(t) = \alpha e^{\alpha t}\left(1 - \frac{t}{\theta}\right)^{-1} + e^{\alpha t}\frac{1}{\theta}\left(1 - \frac{t}{\theta}\right)^{-2}$$

Therefore $E[X] = M_X'(0) = \alpha + \frac{1}{\theta}$. Also

$$M_X''(t) = \alpha^2 e^{\alpha t}\left(1 - \frac{t}{\theta}\right)^{-1} + 2\frac{\alpha}{\theta}e^{\alpha t}\left(1 - \frac{t}{\theta}\right)^{-2} + e^{\alpha t}\frac{2}{\theta^2}\left(1 - \frac{t}{\theta}\right)^{-3}$$

Hence $E[X^2] = M_X''(0) = \alpha^2 + \frac{2\alpha}{\theta} + \frac{2}{\theta^2}$. therefore

$$Var(X) = E[X^2] - (E[X])^2 = \alpha^2 + \frac{2\alpha}{\theta} + \frac{2}{\theta^2} - \left(\alpha + \frac{1}{\theta}\right)^2 = \frac{1}{\theta^2}$$

Part 2

3. (a) By definition of the trinomial distribution we consider n independent trials and $Z = X + Y$ is the number of trials where either "success" or "failure" occurs. Since the probability of {"s" or "f"} in each trial is $p + \theta$ we have that $Z \sim Binomial(n, p + \theta)$.

(b)

$$\begin{aligned}
P(X = x|Z = z) &= \frac{P(X = x, Z = z)}{P(Z = z)} = \frac{P(X = x, Y = z - x)}{P(Z = z)} \\
&= \frac{\frac{n!}{x!(z-x)!(n-z)!} p^x \theta^{z-x} (1 - p - \theta)^{n-z}}{\binom{n}{z} (p + \theta)^z (1 - p - \theta)^{(n-z)}} \\
&= \binom{z}{x} \frac{p^x \theta^{z-x}}{(p + \theta)^z} = \binom{z}{x} \left(\frac{p}{(p + \theta)}\right)^x \left(1 - \frac{p}{(p + \theta)}\right)^{z-x}
\end{aligned}$$

Hence $X|(Z = z) \sim Binomial\left(z, \frac{p}{(p+\theta)}\right)$

4. (a) Use equivalent events. Note that $F_Y(y) = 0$ for $y \leq 0$. For $y > 0$, splitting the range in the integral we obtain,

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) = P(|X| \leq y) = P(-y < X < y) \\
&= \int_{-y}^{y} \frac{\theta}{2} e^{-\theta|x|} dx = \int_{-y}^{0} \frac{\theta}{2} e^{\theta x} dx + \int_{0}^{y} \frac{\theta}{2} e^{-\theta x} dx \\
&= \left[\frac{1}{2} e^{\theta x}\right]_{x=-y}^{x=0} + \left[-\frac{1}{2} e^{-\theta x}\right]_{x=0}^{x=y} = 1 - e^{-\theta y}
\end{aligned}$$

Differentiating the c.d.f. gives $f_Y(y) = \frac{dF_Y(y)}{dy} = \theta e^{-\theta y}$ for $y > 0$ and $f_Y(y) = 0$ elsewhere. Hence $Y \sim Exp(\theta)$.

(b) For $|t| < \theta$,

$$\begin{aligned}
M_X(t) &= E[e^{tX}] = \frac{\theta}{2} \int_{-\infty}^{\infty} e^{tx} e^{-\theta|x|} dx = \frac{\theta}{2} \left(\int_{-\infty}^{0} e^{(\theta+t)x} dx + \int_{0}^{\infty} e^{-(\theta-t)x} dx \right) \\
&= \left[\frac{\theta}{2(\theta + t)} e^{(\theta+t)x}\right]_{x=-\infty}^{x=0} + \left[-\frac{\theta}{2(\theta - t)} e^{-(\theta-t)x}\right]_{x=0}^{x=\infty} \\
&= \frac{\theta}{2(\theta + t)} + \frac{\theta}{2(\theta - t)} = \frac{\theta^2}{\theta^2 - t^2} = \left(1 - \frac{t^2}{\theta^2}\right)^{-1}
\end{aligned}$$

Expanding the m.g.f. in a power series gives

$$M_X(t) = \left(1 - \frac{t^2}{\theta^2}\right)^{-1} = \sum_{r=0}^{\infty} \frac{t^{2r}}{\theta^{2r}}$$

Hence $E[X] = 0$, $E[X^2] = 2!\frac{1}{\theta^2} = \frac{2}{\theta^2}$, $E[X^3] = 0$ and $E[X^4] = 4!\frac{1}{\theta^4} = \frac{24}{\theta^4}$

Hence $\mu = E[X] = 0$, $\sigma^2 = E[(X-\mu)^2] = E[X^2] = \frac{2}{\theta^2}$, $\sqrt{\beta_1} = \frac{E[(X-\mu)^3}{\sigma^3} = \frac{E[X^3]}{\sigma^3} = 0$ and $\beta_2 = \frac{E[(X-\mu)^4}{\sigma^4} = \frac{E[X^4]}{\sigma^4} = \frac{\frac{24}{\theta^4}}{\frac{4}{\theta^4}} = 6$

(c) Since $Z = g(Y) = Y^3$ is a monotone function, we can use the formula explained in Notes 7:

$$f_Z(z) = f_Y(g^{-1}(z)) \left| \frac{dg^{-1}(z)}{dz} \right|$$

(note the trivial change notations!) In our case $g^{-1}(z) = z^{\frac{1}{3}}$. Hence

$$\frac{dg^{-1}(z)}{dz} = \frac{1}{3} z^{-\frac{2}{3}}$$

and thus $f_Z(z) = 0$ if $z < 0$ and for $z > 0$

$$f_Z(z) = f_Y(g^{-1}(z)) \left| \frac{dg^{-1}(z)}{dz} \right| = \theta e^{-\theta z^{\frac{1}{3}}} z^{-\frac{2}{3}}.$$